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ASYMPTOTIC RELATIONS
FOR PARTITIONS

by



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The undersigned certify that they have read, and recommend
to the Faculty of Graduate Studies for acceptance, a thesis entitled
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ABSTRACT

In this thesis we shall be concerned with obtaining asymptotic relations for the number $P_A(n)$ of partitions of the positive integer n into summands (repetitions allowed) from a given set A of positive integers under suitable restrictions on the set A . These restrictions are generally weaker than those in the literature such as those of Ingham [10], Roth and Szekeres [14], and Schwartz [12], for example.

In chapter II we define the function $f(\alpha)$ for real $\alpha > 0$ by the convergent series

$$f(\alpha) = \sum_{a \in A} e^{-\alpha a} .$$

We say that $A = \{a_0, a_1, \dots\}$, where $a_0 < a_1 < \dots$, has property P if either of the conditions (i) or (ii) below hold:

$$(i) \quad s = \lim_{v \rightarrow \infty} \frac{\log \log a_v}{\log v} \text{ exists and is less than unity;}$$

$$(ii) \quad \underline{\lim}_{v \rightarrow \infty} \frac{\log a_v}{v} > 0 .$$

We say that A is a Q -sequence if there does not exist a prime number q such that q/a_v for all $a_v \geq$ some lower bound a_{v_0} .

(ii)

We then derive an asymptotic relation for $P_A(n)$ when A has property P , either A is a Q -sequence or $\log f(\alpha) / \log \frac{1}{\alpha} \rightarrow 0$ as $\alpha \rightarrow 0$, and there exists some δ with $1 > \delta > 0$ such that

$$A\left(\frac{\pi}{\alpha}\right) > f^{\frac{2}{3} + \eta}(\alpha), \quad \eta > 0 \text{ and fixed,}$$

and

$$A\left(\frac{\pi}{\alpha f^{\delta}(\alpha)}\right) / \log f(\alpha) \rightarrow \infty \quad (\alpha \rightarrow 0); \quad \text{where } A(x)$$

is the number of elements of A which are $\leq x$.

Next we show that an asymptotic relation for $P_A(n)$ can be obtained when A satisfies the condition

$$A(2u) \leq C A(u), \quad \text{for } u \geq u_0 \text{ (fixed),}$$

where C is some constant.

In chapter III the general theorems are applied to (i) the case of the Mahler problem ($a_v = r^v$) and (ii) the case when the v -th term of the set A is of the form $a_v = (B^v - B^{-v}) / C$ where B and C are positive constants.

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TABLE OF CONTENTS

	<u>Page</u>
ABSTRACT	(i)
ACKNOWLEDGEMENTS	(ii)
CHAPTER I: Introduction	1
CHAPTER II: Asymptotic Relations for Partitions	13
§2.1 The Basic Theorem and an Outline of its Proof . . .	13
§2.2 The Asymptotic Estimation of I_1	20
§2.3 The Estimation of the Integrals I_2 and I_3 . . .	43
§2.4 Further Asymptotic Relations for Partitions . . .	61
CHAPTER III: Some Applications	71
§3.1 Mahler's Problem	71
§3.2 Partitions into Numbers Related to the Fibonacci Numbers	84
BIBLIOGRAPHY	106

CHAPTER I

Introduction

The problem with which we are concerned in this thesis is that of obtaining asymptotic relations for the number of partitions of a positive integer n into summands from a given set A of positive integers, that is, the number of ways $P_A(n)$ that n can be written in the form

$$n = a_{\ell_1} + a_{\ell_2} + \dots + a_{\ell_m}, \quad a_{\ell_j} \in A.$$

It is well-known [see, for example, [1], pp. 220-222) that the generating function for $P_A(n)$ defined by

$$F_A(x) = \sum_{n=0}^{\infty} P_A(n) x^n$$

has radius of convergence unity and that

$$(1.1) \quad F_A(x) = \prod_{a \in A} (1-x^a)^{-1}.$$

In this thesis we shall first obtain, using the saddle-point method, some asymptotic relations for $P_A(n)$ under some fairly weak conditions on A - probably these conditions are weaker than those imposed by other authors in the past, such as Roth and Szekeres [14],

Schwartz [7], [12], [17], Ingham [10], Pennington [11], de Bruijn [16], etc. Yet the theorems we get on the asymptotic expansion of $P_A(n)$ (theorems (2.4.3) and (2.4.8) in this thesis are strong enough to cover the results of Roth and Szekeres [14], Schwarz [7], [12], [17], Ingham [10], etc. Our results are also applicable to the Mahler problem (where the set A consists of the nonnegative powers of the positive integer $r > 1$) and we obtain, as special cases of our theorems, those results that were obtained by Mahler, de Bruijn, Pennington and Schwarz by other methods. In fact we indicate how following our method we can obtain even better order results than those of de Bruijn on the Mahler problem, though the calculations involved are extremely tedious. We should note that the results of Roth and Szekeres cannot be applied to Mahler's problem.

Among other applications of our general formulae, we give an asymptotic formula (see (3.2.29) in the thesis) for the partition function $P_A(n)$ when A consists of numbers related to the Fibonacci numbers.

In this thesis we do not make use of the celebrated Hardy-Ramanujan circle method [2], but rely entirely on the so-called saddle point method. It may be mentioned here that for the kind of problems treated in this thesis, previous authors mainly utilized two approaches, namely the saddle-point method and Tauberian arguments. In what follows we shall give a brief account of some of the relevant results in the literature.

We begin with a Tauberian theorem of Hardy and Ramanujan which they [3] and later on Brigham [4] applied to several partition problems. In view of its historical interest we shall quote it below:

(1.2) Theorem: Suppose that

$$(1) \quad \lambda_1 \geq 0, \quad \lambda_n \geq \lambda_{n-1}, \quad \lambda_n \rightarrow \infty;$$

$$(2) \quad \lambda_n / \lambda_{n+1} \rightarrow 1;$$

$$(3) \quad a_n \geq 0;$$

$$(4) \quad A > 0, \quad \alpha > 0;$$

$$(5) \quad \sum a_n e^{-\lambda_n s} \text{ is convergent for } s > 0;$$

$$(6) \quad f(s) = \sum a_n e^{-\lambda_n s} = \exp [1+o(1)] A s^{-\alpha} \left\{ \log \frac{1}{s} \right\}^{-\beta},$$

when $s \rightarrow 0$. Then

$$A_n = a_1 + a_2 + \dots + a_n = \exp [\{1+o(1)\} \beta \lambda_n^{\frac{\alpha}{(1+\alpha)}} (\log \lambda_n)^{\frac{-\beta}{(1+\alpha)}}],$$

when $n \rightarrow \infty$, where

$$\beta = A^{1/(1+\alpha)} \alpha^{-\alpha(1+\alpha)} (1+\alpha)^{1+[\beta/(1+\alpha)]}.$$

This Tauberian theorem has been generalized by Kohlbecker [5] and later by Parameswaran [6], among others. When applied to the generating function F_A these Tauberian theorems give asymptotic formulae for $\log P_A(n)$; however, they do not estimate the error term. Recently Schwartz [7] obtained a Tauberian theorem which is more general than most of those obtained before and which gives an estimate of the error term. His theorem is along the lines of Wagner [8], however in a more suitable form for partition theory. Schwarz's theorem is as follows:

Let $A(u)$ be a monotone increasing realvalued function with $A(0) = 0$; let the integral

$$f(\sigma) = \sigma \int_0^{\infty} A(u) \exp(-u\sigma) du$$

be convergent for $\sigma > 0$, and as $\sigma \rightarrow 0$ let

$$\log f(\sigma) = \phi(\sigma) + o(1) ,$$

where the twice continuously differentiable function ϕ satisfies the following conditions:

$$\phi''(\sigma) > 0 , \quad \phi''(\sigma) \uparrow \infty \quad \text{as } \sigma \downarrow 0 ;$$

$$-\sigma\phi'(\sigma) \uparrow \infty \quad \text{as } \sigma \downarrow 0 ;$$

there exists constants $\rho \geq 0$ and C_1 , so that

$$\frac{\sigma(\phi''(\sigma))^{\rho+1}}{|\phi'(\sigma)|^{\rho+2}} \leq C_1 .$$

Then as $T \rightarrow \infty$

$$\log A(T) = \phi(\sigma_T) + T \sigma_T + o\left\{T \sigma_T \left(\frac{\phi''(\sigma)}{|\phi'(\sigma_T)|^2} \cdot \log \frac{|\phi'(\sigma_T)|^2}{\phi''(\sigma_T)}\right)^{1/2}\right\} ,$$

where σ_T is for large T uniquely determined by

$$-\phi'(\sigma_T) = T \quad .$$

We would like to emphasize that none of the Tauberian theorems referred to above use complex function theory methods.

Next we mention the following Tauberian theorem of Ingham which he applied to a wide class of partition problems. However, his theorem makes use of complex variable theory.

Let there be given two functions $\phi(s)$ and $\chi(s)$, and a domain D of the s -plane, satisfying the following conditions:

(a) $\phi(s)$ and $\chi(s)$ are regular in D , and real and positive in a segment $(0, h]$ of the positive real axis lying in D ;

(b) $-\sigma \phi'(\sigma) \uparrow \infty$ as $\sigma \downarrow 0$, [where $\{\sigma \phi'(\sigma)\}' \geq 0$ and therefore

$$(1) \quad \sigma \phi''(\sigma) \geq -\phi'(\sigma) > 0$$

for sufficiently small positive σ];

(c) $\frac{\delta(\sigma)}{\sigma} / \frac{\{\phi''(\sigma)\}^{1/2}}{-\phi'(\sigma)} \rightarrow \infty$ as $\sigma \rightarrow 0$, where $\delta(\sigma)$ is the distance of the point σ from the complement of the set D [so that

$$(2) \quad \delta(\sigma) \leq \sigma, \quad (0 < \sigma \leq h),$$

since the origin cannot belong to D by (a) and (b);

$$(d) \quad \phi''(\sigma+z) = o\{\phi''(\sigma)\} \quad , \\ \chi(\sigma+z) = o\{\chi(\sigma)\} \quad ,$$

uniformly for $|z| < \delta(\sigma)$ when $\sigma \downarrow 0$. Define

$$f_o(s) = \chi(s) e^{\phi(s)} \quad , \quad F_o(s) = f_o(s)/s \quad , \\ A_o(\omega) = \chi(\sigma) \frac{e^{\phi(\sigma) + \omega\sigma}}{\{2\pi \sigma^2 \phi''(\sigma)\}^{1/2}} = \frac{F_o(\sigma) e^{\omega\sigma}}{\{2\pi \phi''(\sigma)\}^{1/2}} \quad ,$$

where $\sigma = \sigma_\omega$ is the solution [which exists and is unique for sufficiently large ω , by (a) and (b)] of

$$(3) \quad -\phi'(\sigma) = \omega \quad , \quad (0 < \sigma \leq h) \quad .$$

In these circumstances we have

(1.3) Theorem: Suppose that

$$f(s) = \int_0^\infty e^{-us} dA(u) \quad , \quad A(0) = 0 \quad , \quad s = \sigma + it$$

is convergent for $\sigma > 0$, and that

$$(i) \quad f(s) \sim f_o(s) \quad \text{when } s \rightarrow 0 \text{ in } D \quad ;$$

(ii) $f(s) = o\{f_o(|s|)\}$ when $s \rightarrow 0$ in some fixed angle ' Δ ' of the form $|t| \leq \Delta\sigma$ ($0 < \Delta < \infty$) .

$$(iii) \quad A(u) \text{ is increasing (in the wide sense) for } u \geq 0 \quad .$$

Then

$$P(\Delta) \leq \overline{\lim}_{u \rightarrow \infty} \frac{A(u)}{A_0(u)} \leq B(\Delta) ,$$

where $P(\Delta)$ and $B(\Delta)$ depend only upon Δ , are strictly positive and finite for any given Δ , ($0 < \Delta < \infty$), and tend to 1 when $\Delta \rightarrow \infty$.

If (ii) holds for every fixed Δ , then

$$A(u) \sim A_0(u) \quad \text{as } u \rightarrow \infty .$$

Asymptotic formulae for $P_A(n)$ have been obtained in this way by Ingham [10], later by Pennington [11], and more recently Schwarz [12].

Another useful method for studying the asymptotic behaviour of some partition problems is the saddle-point method. This method, when applicable, has the advantage of giving generally stronger results. However as yet it has not been applied to many problems other than that of $P_A(n)$, as the previous methods have. G. Meinardus [13] and later Roth and Szekeres [14], applied this technique to $P_A(n)$. They proved the theorem which we give below:

(1.4) Theorem: Let

$$(i) \quad s = \lim_{k \rightarrow \infty} \frac{\log a_k}{\log k} \quad \text{exist,}$$

$$(ii) \quad J_k = \inf_{\frac{1}{2} a_k^{-1} < \alpha \leq \frac{1}{2}} \{ (\log k)^{-1} \sum_{v=1}^k \left(\frac{a_k}{a_v} \right)^2 ||\alpha a_v||^2 \} \rightarrow \infty$$

as $k \rightarrow \infty$.

Then with fixed integral $m \geq 3$ and any constant $\delta > 0$

$$P_A(n) = (2\pi A_2)^{-\frac{1}{2}} \exp \left[\sum_{a \in A} \left\{ \frac{\eta a}{e^{\eta a} - 1} - \log(1 - e^{-\eta a}) \right\} \right] \times \\ \times \left\{ 1 + \sum_{\rho=1}^{m-2} D_\rho + O(n^{-(m-1)}(s+1)^{-1} + \delta) \right\}$$

where η is determined from

$$n = \sum_{a \in A} a(e^{\eta a} - 1)^{-1},$$

and D_ρ ($\rho=1, 2, \dots, m-1$) is given by

$$D_\rho = A_2^{-6\rho} \sum_{\mu_1=2}^{\infty} \dots \sum_{\mu_{5\rho}=2}^{\infty} d_{\mu_1 \mu_2 \dots \mu_{5\rho}} A_{\mu_1} A_{\mu_2} \dots A_{\mu_{5\rho}},$$

the summation being subject to

$$\mu_1 + \mu_2 + \dots + \mu_{5\rho} = 12\rho,$$

where the d 's are certain numerical constants and

$$A_\mu = A_\mu(n) = \sum_{a \in A} a^\mu g_\mu(e^{\eta a})(e^{\eta a} - 1)^{-\mu}.$$

Here $g_\mu(x)$ is a certain polynomial of degree less than μ and, in particular,

$$g_1(x) = 1, \quad g_2(x) = x.$$

In chapter II we shall prove a generalization of this theorem for which condition (ii) of Szekeres and Roth's theorem is not required, and furthermore condition (i) is relaxed. Using this theorem we shall prove generalizations of Schwarz's theorems in [12]. We can show that as long as

$$N(2u) = o\{N(u)\} \quad , \quad \text{as } u \rightarrow \infty$$

where $N(u)$ denotes the number of elements of A less than or equal to u , then an asymptotic formula for $P_A(n)$ may be determined.

In chapter III we give some applications of our general theorems proved in chapter II. Our first application is to the so-called "Mahler's problem". In Mahler's problem, A is the set of powers of a fixed integer $r \geq 2$. Mahler [15] obtained the first results on this problem by applying a theorem on the solution of the functional equation

$$\frac{f(z+\omega) - f(z)}{\omega} = f(z) \quad .$$

He showed that

$$\begin{aligned} \log P_A(rm) &= \frac{1}{2 \log r} (\log(m/\log m))^2 + \\ &+ \left(\frac{1}{2} + \frac{1}{\log r} + \frac{\log \log r}{\log r} \right) \log m \\ &- \left(1 + \frac{\log \log r}{\log r} \right) \log \log m + o(1) \quad , \end{aligned}$$

(where $A = \{1, r, r^2, \dots\}$) . Since it is easily seen that

$$P_A(rm) = P_A(rm + 1) = \dots = P_A(rm + r - 1) ,$$

this gives an asymptotic relation for $P_A(n)$. Next, de Bruijn [16] showed, using the saddle-point method, that the 0-term is actually of the form

$$U\left(\frac{\log m - \log \log m + \log \log r}{\log r}\right) + \log \log r - \frac{1}{2} \log 2\pi + O\left(\frac{(\log \log m)^2}{\log m}\right) ,$$

where

$$U(x) = \sum_{k=-\infty}^{\infty} \alpha_k e^{2\pi i kx} ,$$

and

$$\alpha_k = \begin{cases} \left\{ \gamma_2 - \frac{1}{2} \gamma^2 + \frac{1}{12} \pi^2 + \frac{1}{12} (\log r)^2 / \log r \right. & (k=0) \\ \left. r^{\left(\frac{2\pi i k}{\log r}\right)} \zeta\left(1 + \frac{2\pi i k}{\log r}\right) / \log r \right. & (k=\pm 1, \pm 2, \dots) \end{cases}$$

γ being Euler's constant and γ_2 is defined by

$$z \zeta(z+1) = 1 + \gamma z + \gamma_2 z^2 + \dots .$$

Pennington [11] later derived de Bruijn's result without the error-term estimate. Recently Schwarz [17] obtained Pennington's result.

In chapter III we derive de Bruijn's result as an application of our general results and show that the error-term can be investigated

further without difficulty in principle, although the calculations are extremely lengthy. As a further application, we shall obtain an asymptotic relation for the number of partitions of n into numbers related to the Fibonacci, that is when the n th term of the set A is of the form $\frac{B^n - B^{-n}}{C}$ where B and C are positive constants.

In this thesis we shall primarily be concerned with infinite A 's. However, as suggested by Professor Cheema, let us for the sake of completeness consider briefly the case when A is finite. This case is of course easier to handle than the one when A is infinite. For instance we have the following result (see [22]):

(1.5) Theorem: Let A have r distinct elements $a_1, a_2, \dots, a_r$ and let the elements of A have greatest common divisor unity. Then

$$P_A(n) = \frac{n^{r-1}}{(r-1)! a_1 a_2 \cdots a_r} + O(n^{r-2}) \quad (n \rightarrow \infty).$$

The proof of the above theorem is based on the methods of Cayley, Glaisher, and Sylvester (see [24], and [25]) and is as follows.

By (1.1)

$$\begin{aligned} (1.6) \quad \frac{1}{F_A(x)} &= (1-x)^r \prod_{m=1}^r (1 + x + x^2 + \cdots + x^{a_m-1}) \\ &= (1-x)^r \prod_{m=1}^r \prod_{\ell=1}^{a_m-1} (1 - x e^{2\pi i \ell / a_m}). \end{aligned}$$

Thus if we decompose $F_A(x)$ into partial fractions, the terms in $1-x$ are of the form

$$(1.7) \quad \frac{\alpha_1}{(1-x)} + \frac{\alpha_2}{(1-x)^2} + \cdots + \frac{\alpha_r}{(1-x)^r} ,$$

where $\alpha_r = (a_1 a_2 \dots a_r)^{-1}$. Let $g(n)$ be the coefficient of x^n in (1.7), then

$$g(n) = \sum_{n=1}^r \alpha_n \binom{n+h-1}{h-1} ,$$

which is a polynomial of degree $r-1$ and highest coefficient

$[(r-1)! a_1 a_2 \dots a_r]^{-1}$. If $d > 1$ and ζ is a primitive d th root of unity, the multiplicity of the factor $1 - \zeta x$ in the factorization (1.6) is equal to the number q_d of multiples of d among $a_1, a_2, \dots, a_r$. Since $q_d \leq r-1$ the terms in $1 - \zeta x$ in the decomposition of $F_A(x)$ into partial fractions have the form

$$(1.8) \quad \frac{\beta_1}{(1-\zeta x)} + \frac{\beta_2}{(1-\zeta x)^2} + \cdots + \frac{\beta_{r-1}}{(1-\zeta x)^{r-1}} .$$

The coefficient of $(\zeta x)^n$ in (1.8) is a polynomial of degree at most $r-2$. Summing over all possible ζ , we obtain theorem (1.5). For a more general result in the same direction, we refer to Bateman and Erdős [23].

CHAPTER II

§2.1 Definitions and notations.

Throughout this chapter we use the following definitions and notations.

(2.1.1) $A = \{a_0, a_1, a_2, \dots\}$ is an infinite sequence of integers with $0 < a_0 < a_1 < \dots$.

(2.1.2) Definition: We say that the sequence A is a Q -sequence if there does not exist a prime number q such that q/a_v for all $a_v \geq$ some lower bound a_{v_0} .

(2.1.3) $A(x)$ is the enumerating function of the $\{a_v\}$, that is,

$$(2.1.4) \quad A(x) = \sum_{a_j \leq x} 1.$$

(2.1.5) Definition: For all real $\alpha > 0$, we define $f(\alpha)$ by the relation

$$f(\alpha) = \sum_{v=0}^{\infty} e^{-\alpha a_v},$$

we note that since the a_v are positive integers the series on the right is convergent for all $\alpha > 0$.

(2.1.6) Definition:

$$\kappa_1(A) = \lim_{v \rightarrow \infty} \frac{\log \log a_v}{\log v}$$

$$\kappa_2(A) = \lim_{v \rightarrow \infty} \frac{\log \log a_v}{\log v}$$

(2.1.7) Definition: We say that A has property P if any one of the properties (2.1.8) or (2.1.9) below holds:

$$(2.1.8) \quad 0 \leq \kappa_1(A) = \kappa_2(A) < 1$$

$$(2.1.9) \quad \lim_{v \rightarrow \infty} \frac{\log a_v}{v} > 0 .$$

(2.1.10) Remark: We note that the case (2.1.9) occurs only when $\kappa_1(A) \geq 1$.

(2.1.11) Remark: If A is a sequence such that $\kappa_2(A) < 1$ then it is easy to see that

$$\lim_{v \rightarrow \infty} \frac{\log a_v}{v} = 0 .$$

Thus we may say A has property P if either $\lim_{v \rightarrow \infty} \frac{\log a_v}{v} = 0$ and $\lim_{v \rightarrow \infty} \frac{\log \log a_v}{\log v}$ exists and is less than one or $\lim_{v \rightarrow \infty} \frac{\log a_v}{v} > 0$.

(2.1.12) The basic theorem and the plan of its proof: We shall now prove the following basic theorem and later give some extensions. This theorem is fundamental for much of our later work. Before stating our theorem however we note that since $a_v \geq v$,

$$f(\alpha) \leq \sum_{v=0}^{\infty} e^{-\alpha v} \leq \int_{-1}^{\infty} e^{-\alpha x} dx = O\left(\frac{1}{\alpha}\right) \quad (\alpha \rightarrow 0) .$$

Thus

$$\overline{\lim}_{\alpha \rightarrow 0} \frac{\log f(\alpha)}{\log \frac{1}{\alpha}} \leq 1 .$$

(2.1.13) Theorem: Let A be a sequence having either of the following properties:

$$(2.1.14) \quad \lim_{\alpha \rightarrow 0} \log f(\alpha) / \log \frac{1}{\alpha} = 0 .$$

(2.1.15) A is a Q-sequence.

Let A have property P .

Assume furthermore that for some constant δ with $1 > \delta > 0$

$$\frac{A\left(\frac{\pi}{\alpha^\delta}\right)}{\log f(\alpha)} \rightarrow \infty \quad (\alpha \rightarrow 0)$$

and

$$A\left(\frac{\pi}{\alpha}\right) > f^{\frac{2}{3} + \eta}(\alpha)$$

where η is some constant with $\frac{1}{3} \geq \eta > 0$. Let m be any fixed integer ≥ 3 . Then

$$P_A(n) = (2\pi A_2)^{-\frac{1}{2}} \exp \left\{ \sum_{v=0}^{\infty} \left[\frac{\alpha a_v}{1 - e^{-\alpha a_v}} + \log (1 - e^{-\alpha a_v}) \right] \right\} \times \\ \times \left[1 + \sum_{\rho=1}^{m-2} D_{\rho} + O \left\{ e^{-\frac{2m}{3}(\alpha)} \right\} \right],$$

(the constants implied by the O -terms depend upon m but not α)

where $\alpha = \alpha(n)$ is determined from

$$n = \sum_{v=0}^{\infty} \frac{a_v}{e^{\alpha a_v} - 1},$$

and D_{ρ} ($\rho=1,2,\dots,m-2$) is determined from

$$D_{\rho} = A_2^{-6\rho} \sum_{\mu_m=2}^{\infty} \cdots \sum_{\mu_{5\rho}=2}^{\infty} d_{\mu_1 \mu_2 \cdots \mu_{5\rho}} A_{\mu_1} A_{\mu_2} \cdots A_{\mu_{5\rho}}$$

the summation being subject to

$$\mu_1 + \mu_2 + \cdots + \mu_5 = 12\rho,$$

where the d 's are certain numerical constants and

$$A_{\mu} = A_{\mu}(n) = \sum_{v=0}^{\infty} a_v^{\mu} g_{\mu} (e^{\alpha a_v}) (e^{\alpha a_v} - 1)^{-\mu},$$

and in particular, $g_1(x) = 1$ and $g_2(x) = x$.

(2.1.16) Remark: Before proceeding to the proof of this theorem, let us remark that the conditions

$$A\left(\frac{\pi}{\alpha f^\delta(\alpha)}\right) / \log f(\alpha) \rightarrow \infty$$

and

$$A\left(\frac{\pi}{\alpha}\right) > f^{\frac{2}{3} + \eta}(\alpha)$$

are satisfied by most commonly occurring sequences. For example, it can be shown that they hold when

$$A(2u) = O\{A(u)\} \quad ;$$

or when

$$S = \lim_{v \rightarrow \infty} \frac{\log a_v}{\log v}$$

exists. (See remark (2.4.4), lemmas (2.4.5) and (2.4.6) for a detailed proof of this statement.

Proof: The proof of the theorem is very long. We shall give the plan of the proof in the rest of this section, and the various details of the same in the next two sections.

We recall from chapter I that the generating function $F_A(z)$ for $P_A(n)$ (with $P_A(0) = 1$) is:

$$F_A(z) = \sum_{n=0}^{\infty} P_A(n) z^n = \prod_{n=0}^{\infty} (1 - z^{a_n})^{-1},$$

where the series and product above converge for $|z| < 1$. From Cauchy's theorem,

$$(2.1.17) \quad P_A(n) = \frac{1}{2\pi i} \oint F_A(\xi) \xi^{-(n+1)} d\xi$$

where the path of integration is taken to be a circle with centre at the origin and radius $\rho < 1$. We choose ρ to be the positive value for which the logarithmic derivative of the integrand vanishes. This will be a saddle-point of the surface formed by the absolute values of the integrand in (2.1.17). This saddle-point condition is

$$n+1 = \rho \frac{F'_A(\rho)}{F_A(\rho)} = \sum_{v=0}^{\infty} \frac{a_v \rho^{a_v}}{1 - e^{a_v \rho}}.$$

If we let $\rho = e^{-\alpha}$, $\alpha > 0$ and replace $n+1$ by n to make the condition somewhat simpler, it becomes

$$(2.1.18) \quad n = \sum_{v=0}^{\infty} a_v (e^{\alpha a_v} - 1)^{-1}.$$

Note that this can be solved uniquely for α and hence ρ , since the sum is a monotonic function of α which takes on all positive real values. We now proceed to estimate the integral in (2.1.17).

Letting $\xi = \rho e^{i\theta}$ in (2.1.17) we obtain

$$\begin{aligned} P_A(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} F_A(\rho e^{i\theta}) \rho^{-n} e^{-ni\theta} d\theta \\ &= \frac{1}{2\pi} \exp \{n\alpha + \log F_A(\rho)\} \int_{-\pi}^{\pi} \frac{F_A(\rho e^{i\theta})}{F_A(\rho)} e^{-ni\theta} d\theta . \end{aligned}$$

Hence, using (2.1.18), we have

$$\begin{aligned} P_A(n) &= \frac{1}{2\pi} \exp \left\{ \sum_{v=0}^{\infty} [\alpha a_v (e^{\alpha a_v} - 1)^{-1} - \log (1 - e^{-\alpha a_v})] \right\} \times \\ &\times \int_{-\pi}^{\pi} \exp \left\{ - \sum_{v=0}^{\infty} \log \left(\frac{1 - \exp(-\alpha a_v + i a_v \theta)}{1 - \exp(-\alpha a_v)} \right) - ni\theta \right\} d\theta . \end{aligned}$$

We dissect the integration interval into 3 parts:

$$(2.1.19) \quad \int_{-\pi}^{\pi} = I_1 + I_2 + I_3 , \quad I_1 = \int_{-\theta_0}^{\theta_0} , \quad I_2 = \int_{\theta_0}^{\pi} , \quad I_3 = \int_{-\pi}^{-\theta_0} .$$

The value of θ_0 is important. θ_0 must be small enough so that we can approximate the integrand in I_1 by its Taylor series expansion in θ . On the other hand the contributions to $P_A(n)$ due to I_2 and I_3 must be negligible, thus θ_0 cannot be too small. As we shall see later in the next two sections, the value of θ_0 given by

$$(2.1.20) \quad \theta_0 = \alpha (f(\alpha))^{-\left(\frac{1}{3} + \frac{\eta}{3}\right)}$$

is in fact the right choice, where η is that of theorem (2.1.13). In

the subsequent discussion, we shall assume that θ_0 has this value.

In §2.2 we shall show that for sets A which have property B with this choice of θ_0 , I_1 has the claimed asymptotic expansion of $P_A(n)$ in theorem (2.1.13). In §2.3 moreover it will be shown that I_2 and I_3 have negligible orders of magnitude as indicated.

§2.2 The Asymptotic Estimation of I_1 .

In this section we shall obtain the asymptotic expansion of I_1 where $\theta_0 = \alpha f_{(\alpha)}^{-1/3} - \eta/3$. Our method is a modification of that of Roth and Szekeres [14]; however we shall relax the restrictions on A . We estimate the integrand in I_1 by using its Taylor series. Let

$$\begin{aligned}
 (2.2.1) \quad f_v(\theta) &= -\log \left(\frac{1 - e^{-\alpha a_v + i a_v \theta}}{1 - e^{-\alpha a_v}} \right) \\
 &= -\log \left(1 - \frac{e^{i a_v \theta} - 1}{e^{\alpha a_v} - 1} \right) \\
 &= \sum_{\ell=1}^{\infty} \frac{1}{\ell} \frac{1}{(e^{\alpha a_v} - 1)^{\ell}} \left\{ \sum_{k=1}^{\infty} \frac{(i \theta a_v)^k}{k!} \right\}^{\ell}.
 \end{aligned}$$

This double series will converge if $\theta_0 \leq \frac{\alpha}{2}$ since

$$\sum_{\ell=1}^{\infty} \frac{1}{\ell} (e^{\alpha a_v} - 1)^{-\ell} \left\{ \sum_k \frac{1}{k!} |\theta a_v|^k \right\}^{\ell}$$

is the formal expansion of

$$- \log \{1 - (e^{\frac{|\theta| a_v}{2}} - 1)(e^{\alpha a_v} - 1)^{-1}\}$$

and

$$\frac{(e^{\frac{|\theta| a_v}{2}} - 1)}{e^{\alpha a_v} - 1} \leq \frac{e^{\frac{\alpha}{2} a_v} - 1}{e^{\alpha a_v} - 1} = \frac{1}{\frac{e^{\alpha a_v}}{e^{\frac{\alpha}{2} a_v}} + 1} \leq \frac{1}{2} .$$

Clearly if α is sufficiently small (say, $\alpha < \alpha_0$, where α_0 is fixed), we have $\theta_0 < \frac{\alpha}{2}$ by (2.1.20), since $f(\alpha) \rightarrow \infty$ as $\alpha \rightarrow 0$. We may therefore write

$$\begin{aligned} f_v(\theta) &= \sum_{\ell=1}^{\infty} \frac{1}{\ell} (e^{\alpha a_v} - 1)^{\ell} \left\{ \sum_{k=1}^{\infty} \frac{(i \theta a_v)^k}{k!} \right\}^{\ell} \\ &= \sum_{\mu=1}^{\infty} \frac{b_{\mu}}{\mu!} (i \theta a_v)^{\mu} \end{aligned}$$

where

$$b_{\mu} = \sum_{\ell=1}^{\mu} \underbrace{\sum_{k_1=1} \cdots \sum_{k_{\ell}=1}}_{k_1 + k_2 + \cdots + k_{\ell} = \mu} \frac{(e^{\alpha a_v} - 1)^{-\ell}}{k_1! k_2! \cdots k_{\ell}!} .$$

Thus

$$(2.2.2) \quad b_{\mu} = (e^{\alpha a_v} - 1)^{-\mu} g_{\mu}(e^{\alpha a_v}) ,$$

where g_μ is a polynomial of degree not exceeding $\mu-1$ and in particular;

$$g_1(x) = 1, \quad g_2(x) = x.$$

Now

$$(2.2.3) \quad f_v(\theta) = \sum_{\ell=1}^{\infty} \frac{1}{\ell} (e^{\alpha a_v} - 1)^{-\ell} (e^z - 1)^\ell, \quad z = i \theta a_v.$$

Suppose with integral $m \geq 2$ that

$$\sum_{\ell=1}^{2m-1} \frac{1}{\ell} (e^{\alpha a_v} - 1)^{-\ell} (e^z - 1)^\ell = \sum_{\mu=1}^{\infty} \frac{1}{\mu!} b_\mu^{(1)} z^\mu$$

$$\sum_{\ell=2m}^{\infty} \frac{1}{\ell} (e^{\alpha a_v} - 1)^{-\ell} (e^z - 1)^\ell = \sum_{\mu=1}^{\infty} \frac{1}{\mu!} b_\mu^{(2)} z^\mu,$$

so that $b_\mu^{(1)} + b_\mu^{(2)} = b_\mu$.

Now since the coefficients in the expansion of $(e^z - 1)^\ell$ do not exceed the corresponding coefficients in the expansion of $e^{z\ell}$:

$$\begin{aligned} \left| \sum_{\mu=2m}^{\infty} \frac{1}{\mu!} b_\mu^{(1)} z^\mu \right| &\leq \sum_{\ell=1}^{2m-1} \frac{1}{\ell} (e^{\alpha a_v} - 1)^{-\ell} \sum_{k=2m}^{\infty} \frac{1}{k!} \ell^k |z|^k \\ &= O \left\{ \sum_{\ell=1}^{2m-1} \frac{1}{\ell} (e^{\alpha a_v} - 1)^{-\ell} |z|^{2m} e^{\ell|z|} \right\} \\ &= O \left\{ |z|^{2m} \sum_{\ell=1}^{2m-1} \frac{1}{\ell} \left(\frac{e^{|z|}}{e^{\alpha a_v} - 1} \right)^\ell \right\} \end{aligned}$$

$$= 0 \left\{ |z|^{2m} \sum_{\ell=1}^{2m-1} \frac{1}{\ell} (e^{\beta a_v} - 1)^\ell \right\}$$

where the 0-constant is independent of α and v and

$$\beta = \alpha \left(1 - f - \frac{1}{3} - \frac{\eta}{3}(\alpha) \right).$$

(Recall that $z = i \theta a_v$ by (2.2.3) thus $z = 0(\theta_o a_v)$.

Also

$$\begin{aligned} \left| \sum_{\mu=2m}^{\infty} \frac{1}{\mu!} b_{\mu}^{(2)} z^{\mu} \right| &\leq \sum_{\ell=2m}^{\infty} \frac{1}{\ell} \left(\frac{e^{|z|} - 1}{e^{\alpha a_v} - 1} \right)^{\ell} \\ &= 0 \left\{ \sum_{\ell=2m}^{\infty} (e^{|z|} - 1)^{\ell} (e^{\alpha a_v} - 1)^{-\ell} \right\} \\ &= 0 \left\{ |z|^{2m} \left(e^{\beta a_v} - 1 \right)^{2m} \right\} \end{aligned}$$

where the 0-constant is again independent of α and v . Thus

$$\begin{aligned} \sum_{v=0}^{\infty} f_v(\theta) &= \sum_{v=0}^{\infty} \left(\sum_{\mu=0}^{2m-1} \frac{b_{\mu}}{\mu!} (i \theta a_v)^{\mu} \right) + \\ (2.2.4) \quad &+ 0 \left(\sum_{v=0}^{v_o} \frac{|z|^{2m}}{(e^{\beta a_v} - 1)^{2m-1}} \right) + 0 \left(\sum_{v=v_o+1}^{\infty} \frac{|z|^{2m}}{(e^{\beta a_v} - 1)} \right) \\ &+ 0 \left(\sum_{v=0}^{\infty} |z|^{2m} (e^{\beta a_v} - 1)^{-2m} \right), \end{aligned}$$

where v_o is chosen so that $e^{\beta a_{v_o}} - 1 \leq 1 < e^{\beta a_{v_o+1}} - 1$. We finally obtain that

$$\begin{aligned}
 \sum_{v=0}^{\infty} f_v(\theta) &= \sum_{v=0}^{\infty} \left(\sum_{\mu=1}^{2m-1} \frac{b_{\mu}}{\mu!} (i\theta a_v)^{\mu} \right) + \\
 (2.2.5) \quad &+ 0 \left\{ \sum_{v=0}^{\infty} \frac{|z|^{2m}}{(\beta a_v - 1)^{2m-1}} + \sum_{v=0}^{\infty} \frac{|z|^{2m}}{e^{\beta a_v} - 1} + \sum_{v=0}^{\infty} |z|^{2m} \times \right. \\
 &\quad \left. \times (e^{\beta a_v} - 1)^{-2m} \right\}.
 \end{aligned}$$

We need to estimate this 0-term. From (2.2.3) $|z| \leq \alpha a_v$ and $\frac{\beta}{\alpha} \rightarrow 1$ as $\alpha \rightarrow 0$. We therefore now consider sums

$$\sum_{v=0}^{\infty} \frac{(\beta a_v)^{\mu}}{(e^{\beta a_v} - 1)^j}, \quad \mu \geq j, \quad j = 1, 2, \dots, 2m.$$

(2.2.6) Lemma. Let μ and j be integers with $\mu \geq j$. Then

$$(2.2.7) \quad \sum_{v=0}^{\infty} \frac{(\beta a_v)^{\mu}}{(e^{\beta a_v} - 1)^j} = O \left\{ \sum_{v=0}^{\infty} [1 + (\beta a_v)^{\mu}] e^{-\alpha a_v} \right\}$$

where the O-constant is independent of β but depends upon μ and j .

Proof: Let c be some constant > 0 . If $\beta a_v \geq c$

$$\begin{aligned}
 \frac{1}{(e^{\beta a_v} - 1)^j} &= \left(\frac{e^{\beta a_v}}{e^{\beta a_v} - 1} \right)^j e^{-j\beta a_v} \\
 &\leq \left(\frac{1}{1 - e^{-\beta a_v}} \right)^j e^{-\beta a_v} \\
 &\leq \left(\frac{1}{1 - e^{-c}} \right)^j e^{-\beta a_v}.
 \end{aligned}$$

Hence for $\beta a_v \geq c$

$$\frac{(\beta a_v)^\mu}{(e^{\beta a_v} - 1)^j} = O \{ (\beta a_v)^\mu e^{-\beta a_v} \} = O \{ [1 + (\beta a_v)^\mu] e^{-\beta a_v} \} .$$

Now for $\beta a_v < c$

$$\max_{0 < \beta a_v < c} \frac{(\beta a_v)^\mu}{(e^{\beta a_v} - 1)^j} = h < \infty$$

and

$$\min_{0 < \beta a_v < c} e^{-\beta a_v} \geq e^{-c} .$$

Hence for $\beta a_v < c$

$$\frac{(\beta a_v)^\mu}{(e^{\beta a_v} - 1)^j} = O \{ e^{-\alpha a_v} \} = O \{ [1 + (\beta a_v)^\mu] e^{-\alpha a_v} \} .$$

Since the O -constants are independent of β , the lemma is proven.

Lemma (2.2.6) enables us to simplify the sums that we are led to in equation (2.2.5). We next give an estimate for the sums on the right-hand side of (2.2.7). Recall our definition (2.1.6) of $\kappa_2(A)$.

(2.2.8) Theorem: Let $\kappa_2(A) = 0$. Let μ be any fixed integer > 0 and $\epsilon > 0$ an arbitrary constant. Then

$$\sum_{v=0}^{\infty} (\alpha a_v)^\mu e^{-\alpha a_v} = O \{ f^{1+\epsilon}(\alpha) \} ,$$

as $\alpha \rightarrow 0$. The 0-constant depends upon μ and ϵ but not α .

Proof: We first prove that if $\kappa_2(A) = 0$ then

$$f(\alpha) > \frac{e^{-1}}{4} \log^{\psi} \frac{1}{\alpha}$$

as $\alpha \rightarrow 0$ for every constant ψ .

Let δ be an arbitrary constant > 0 . Since $\kappa_2(A) = 0$ we have that

$$a_v < e^{v^{\delta}}$$

for all v 's greater than some lower bound $v_0 = v_0(\delta)$. Let v be the largest element of A such that $\alpha a_v \leq 1$. Then

$$\begin{aligned} f(\alpha) &> \sum_{v=v_0}^v e^{-\alpha a_v} \\ &> e^{-1} [A(\frac{1}{\alpha}) - v_0] > \frac{e^{-1}}{2} A(\frac{1}{\alpha}) \end{aligned}$$

for all α sufficiently small. However $A(\frac{1}{\alpha}) > \frac{1}{2} \log^{1/\delta}(\frac{1}{\alpha})$ for all sufficiently small α . Thus $f(\alpha) > \frac{e^{-1}}{4} \log^{1/\delta} \frac{1}{\alpha}$ as $\alpha \rightarrow 0$.

Let a_v be the largest element of A such that

$$a_v \leq \frac{f^{\frac{\epsilon}{\mu}}(\alpha)}{\alpha}.$$

Then

$$\sum_{v=1+v}^{\infty} (\alpha a_v)^{\mu} e^{-\alpha a_v} \leq \sum_{v=a_{1+v}}^{\infty} (\alpha v)^{\mu} e^{-\alpha v}$$

$$\leq \int_{a_{1+v}}^{\infty} (\alpha x)^{\mu} e^{-\alpha x} dx$$

since $(\alpha v) e^{-\alpha v}$ is decreasing when $\alpha v \geq \alpha a_{1+v} \geq \frac{\epsilon}{f^{\mu}(\alpha)}$ for sufficiently small α . Thus

$$\sum_{v=1+v}^{\infty} (\alpha a_v)^{\mu} e^{-\alpha a_v} \leq \frac{1}{\alpha} \int_{\alpha(a_{1+v}-1)}^{\infty} y^{\mu} e^{-y} dy$$

$$= O\left\{ \frac{1}{\alpha} \int_{\frac{\frac{\epsilon}{f^{\mu}(\alpha)}}{2}}^{\infty} e^{-y} y^{\mu} dy \right\}$$

$$= O\left\{ \frac{1}{\alpha} \int_{\frac{\frac{\epsilon}{f^{\mu}(\alpha)}}{2}}^{\infty} e^{-\frac{y}{2}} dy \right\}$$

$$= O\left\{ \frac{1}{\alpha} e^{-\frac{1}{4} \log \frac{\psi \epsilon}{\mu}} \frac{1}{\alpha} \right\},$$

for each $\psi > 0$. Since we can take $\psi > \frac{\mu}{\epsilon}$ for α small enough this 0-term is $O(1)$.

We now consider

$$\sum_{a_v \leq a_v} (\alpha a_v)^{\mu} e^{-\alpha a_v}.$$

Since $(\alpha a_v)^{\mu} \leq (\alpha a_v)^{\mu} \leq f^{\epsilon}(\alpha)$ we obtain

$$\sum_{a_v < a_v} (\alpha a_v)^\mu e^{-\alpha a_v} \leq f^\epsilon(\alpha) \sum_{a_v < a_v} e^{-\alpha a_v} \\ < f^\epsilon(\alpha) f(\alpha) .$$

This shows that

$$\sum_{v=0}^{\infty} (\alpha a_v)^\mu e^{-\alpha a_v} < f^{1+\epsilon}(\alpha) + o(1) ,$$

thus proves the theorem.

(2.2.9) Theorem: Let $0 < \kappa_1(A) = \kappa_2(A) < 1$. Let μ be any fixed integer > 0 and ϵ any constant > 0 . Then

$$\sum_{v=0}^{\infty} (\alpha a_v)^\mu e^{-\alpha a_v} = o\{f^{1+\epsilon}(\alpha)\} ,$$

as $\alpha \rightarrow 0$. The constant of the 0-term depends upon μ and ϵ but not α .

Proof: Let $\kappa = \kappa_1(A) = \kappa_2(A)$.

For every $\delta > 0$ there is a fixed $v_0 = v_0(\delta)$ such that $v \geq v_0$ implies that

$$e^{v^{\kappa-\delta}} \leq a_v \leq e^{v^{\kappa+\delta}} .$$

Let

$$\delta = \frac{\epsilon \kappa}{4 + \epsilon}$$

We prove that

$$\log f(\alpha) > \frac{1}{\kappa+\delta} \log \log \frac{1}{\alpha} + o(1)$$

by the same reasoning that showed

$$f(\alpha) > \log^{\psi} \frac{1}{\alpha}$$

in the proof of theorem (2.2.8).

Now $y^{\mu} e^{-y}$ is monotonic decreasing for $y \geq c = c(\mu)$.

Let

$$f_{\mu}(\alpha) = \sum_{v=0}^{\infty} (\alpha a_v)^{\mu} e^{-\alpha a_v} = \sum' (\alpha a_v)^{\mu} e^{-\alpha a_v} + \sum'' (\alpha a_v)^{\mu} e^{-\alpha a_v},$$

where $\sum'$ denotes summation over those v for which $\alpha a_v \leq c$, and $\sum''$ denotes summation over those μ for which $\alpha a_v > c$. Clearly

$$\sum' (\alpha a_v)^{\mu} e^{-\alpha a_v} = o(f(\alpha)),$$

thus we may assume that $\sum'' > \sum'$. Since v_0 is fixed, for α sufficiently small

$$\begin{aligned} \sum'' (\alpha a_v)^{\mu} e^{-\alpha a_v} &\leq \sum_{v=v_0+1}^{\infty} (\alpha e^{v^{\kappa-\delta}})^{\mu} e^{-\alpha} e^{v^{\kappa-\delta}} \\ &< \int_{v_0}^{\infty} (\alpha e^{x^{\kappa-\delta}})^{\mu} e^{-\alpha e^{v^{\kappa-\delta}}} dx \\ &\leq \int_{\alpha}^{\infty} y^{\mu-1} e^{-y} \frac{d}{dy} \left(\log^{\frac{1}{\kappa-\delta}} \frac{y}{\alpha} \right) dy. \end{aligned}$$

Now upon integrating by parts

$$\begin{aligned} \int_{\alpha}^{\infty} y^{\mu-1} e^{-y} \frac{d}{dy} \left(\log^{\frac{1}{\kappa-\delta}} \frac{y}{\alpha} \right) dy &= - \int_{\alpha}^{\infty} (\mu-1) y^{\mu-2} e^{-y} \log^{\frac{1}{\kappa-\delta}} \frac{y}{\alpha} dy \\ &+ \int_{\alpha}^{\infty} y^{\mu-1} e^{-y} \log^{\frac{1}{\kappa-\delta}} \frac{y}{\alpha} dy \quad . \end{aligned}$$

Thus

$$\sum'' (\alpha a_v)^{\mu} e^{-\alpha a_v} \leq \int_{\alpha}^{\infty} y^{\mu-1} e^{-y} \log^{\frac{1}{\kappa-\delta}} \frac{y}{\alpha} dy \quad .$$

Since

$$\int_{\alpha}^{e^2} y^{\mu-1} e^{-y} \log^{\frac{1}{\kappa-\delta}} \frac{y}{\alpha} dy < \log^{\frac{1}{\kappa-\delta}} \frac{e^2}{\alpha} \int_{\alpha}^{e^2} y^{\mu-1} e^{-y} dy \quad ,$$

and

$$\int_{e^2}^{\infty} y^{\mu-1} e^{-y} \log^{\frac{1}{\kappa-\delta}} \frac{y}{\alpha} dy < \log^{\frac{1}{\kappa-\delta}} \frac{1}{\alpha} \int_{e^2}^{\infty} y^{\mu-1} e^{-y} \log^{\frac{1}{\kappa-\delta}} y dy \quad ,$$

(because $(\log y + \log \frac{1}{\alpha})^{1/\kappa-\delta} < \log^{1/\kappa-\delta} \frac{1}{\alpha} \log^{1/\kappa-\delta} y$ for $\log y \geq 2$

and $\log \frac{1}{\alpha} \geq 2$) we obtain that

$$\log \sum'' < \frac{1}{\kappa-\delta} \log \log \frac{1}{\alpha} + 0(1) \quad .$$

Since $\sum' \leq \sum''$ this gives that

$$\log f_{\mu}(\alpha) < \frac{1}{\kappa-\delta} \log \log \frac{1}{\alpha} + 0(1) \quad .$$

Since

$$\delta < \frac{\epsilon \kappa}{4+\epsilon}$$

we have

$$\delta < \frac{\frac{\epsilon}{2}}{2 + \frac{\epsilon}{2}} \kappa$$

$$\kappa + \delta + \delta(1 + \frac{\epsilon}{2}) < (1 + \frac{\epsilon}{2}) \kappa$$

$$\frac{1}{\kappa-\delta} < \frac{1 + \frac{\epsilon}{2}}{\kappa+\delta}$$

and we conclude that

$$\begin{aligned} \log f_{\mu}(\alpha) &< \frac{(1 + \frac{\epsilon}{2})}{\kappa+\delta} \log \log \frac{1}{\alpha} + o(1) \\ &< (1+\epsilon) \log f(\alpha) \end{aligned}$$

for all α sufficiently small. This proves theorem (2.2.9).

(2.2.10) Theorem: Let

$$\lim_{v \rightarrow 0} \frac{\log a_v}{v} > 0 .$$

Let μ be any fixed integer > 1 and $\epsilon > 0$ an arbitrary constant.

Then

$$\sum_{v=0}^{\infty} (\alpha a_v)^{\mu} e^{-\alpha a_v} = o\{f(\alpha)\} ,$$

as $\alpha \rightarrow 0$. The o -term constant depends upon μ and ϵ but not α .

Proof: Let

$$\lim_{v \rightarrow \infty} \frac{\log a_v}{v} = c .$$

Then for each $\delta > 0$, there exists a fixed $v_0 = v_0(\delta)$ such that if

$$v \geq v_0$$

$$a_v \geq e^{(c-\delta)v} .$$

Choose

$$\delta = \frac{c}{2} .$$

Now $y^\mu e^{-y}$ is monotonic decreasing for $y > y_0 = y_0(\mu)$. Since

$$\sum_{v=0}^{\infty} (\alpha a_v)^\mu e^{-\alpha a_v} = \sum' (\alpha a_v)^\mu e^{-\alpha a_v} + \sum'' (\alpha a_v)^\mu e^{-\alpha a_v} ,$$

where $\sum''$ denotes summation over those v for which $\alpha a_v \geq y_0$.

Certainly

$$\sum' (\alpha a_v)^\mu e^{-\alpha a_v} = O\{f(\alpha)\} .$$

Since v_0 is fixed, for α sufficiently small

$$\begin{aligned} \sum'' (\alpha a_v)^\mu e^{-\alpha a_v} &\leq \sum_{v=v_0+1}^{\infty} (\alpha e^{(c-\delta)v})^\mu e^{-\alpha e^{(c-\delta)v}} \\ &< \int_{v_0}^{\infty} (\alpha e^{(c-\delta)x})^\mu e^{-\alpha e^{(c-\delta)x}} dx . \end{aligned}$$

If we let $y = \alpha \exp ((c-\delta)x)$ in this integral we obtain

$$\begin{aligned} \sum'' &< \frac{1}{c-\delta} \int_{\alpha v_0}^{\infty} y^{\mu-1} e^{-y} dy \\ &< \frac{2(\mu-1)!}{c} . \end{aligned}$$

(2.2.11) Remark: Theorems (2.2.8), (2.2.9), and (2.2.10) show that

$$\sum_{v=0}^{\infty} (\alpha a_v)^{\mu} e^{-\alpha a_v} = O\{f^{1+\epsilon}(\alpha)\}$$

for sets that have property P . We shall now give an example of a set which does not have property P .

Let

$$a_v = v , v = 1, 2, \dots, 10 .$$

$$a_{10+j} = 10^{10} + j , j = 1, 2, \dots, 10^{10} - 10$$

$$a_{10^{10}+j} = 10^{10^{10}} + j , j = 1, 2, \dots, 10^{10^{10}} - 10^{10} , \text{ etc.}$$

If

$$x_j = 2 \cdot 10^{\overbrace{10 \dots 10}^j} , N(x_j) > \frac{x_j}{4}$$

and if

$$y_j = 10^{\overbrace{10 \dots 10}^j} + 1, \quad N(y_j) < 3 \log y_j.$$

The writer does not know of any sequence for which the conclusions of theorems (2.2.8), (2.2.9) and (2.2.10) do not hold.

(2.2.11) Theorem: Let A have property P . Let m be a fixed integer > 3 and ϵ any constant > 0 . Then

$$\sum_{v=0}^{\infty} f_v(\theta) = \sum_{v=0}^{\infty} \left(\sum_{\mu=1}^{2m-1} \frac{b_{\mu}}{\mu!} (i\theta a_v)^{\mu} \right) + O\left\{ \left(\frac{\theta}{\alpha}\right)^{2m} f^{1+\epsilon}(\alpha) \right\},$$

as $\alpha \rightarrow 0$. The 0-constant depends upon m and ϵ but not α .

Proof: From theorems (2.2.8), (2.2.9), and (2.2.10) we obtain that

$$\begin{aligned} \sum_{v=0}^{\infty} \frac{|z|^{2m}}{\beta a_v e^{a_v - 1}} &= \left(\frac{\theta}{\beta}\right)^{2m} \sum_{v=0}^{\infty} \frac{(\beta a_v)^{2m}}{\beta a_v e^{a_v - 1}} \\ &= O\left\{ \left(\frac{\theta}{\beta}\right)^{2m} f^{1+\epsilon}(\beta) \right\}. \end{aligned}$$

Since $\beta \sim \alpha$ as $\alpha \rightarrow 0$ we have $\beta^{-1} = O(\alpha^{-1})$. Thus we need only show that $f(\beta) = O\{f^{1+\delta}(\alpha)\}$. The other terms in (2.2.5) are treated similarly.

Let us consider the case $\kappa_2(A) = 0$ first. As in the proof of theorem (2.2.8) we have that

$$f(\beta) = \sum_{v=0}^v e^{-\beta a_v} + o(1) ,$$

where v is the largest integer v such that $\beta a_v \leq f^{1/5}(\alpha)$. However for $v \leq v$ by (2.2.3)

$$\begin{aligned} \frac{e^{-\beta a_v}}{e^{-\alpha a_v}} &= e^{-\alpha f^{-\frac{2}{5}}(\alpha) a_v} , \\ &\leq e^{-f^{-\frac{1}{5}}(\alpha)} = o(1) . \end{aligned}$$

Thus $\sum_{v=0}^v e^{-\beta a_v} = o\left(\sum_{v=0}^v e^{-\alpha a_v}\right)$, and since

$$\sum_{v=v}^{\infty} e^{-\alpha a_v} = o(1)$$

we have $f(\beta) = f(\alpha) + o(1)$.

Next we suppose that

$$0 < \kappa_1(A) = \kappa_2(A) < 1 .$$

Then the arguments used to prove theorem (2.2.9) show that

$$\log f(\alpha) \sim \frac{1}{\kappa_1(A)} \log \log \frac{1}{\alpha}$$

as $\alpha \rightarrow 0$.

and $\log f(\beta) \sim \frac{1}{\kappa} \log \log \frac{1}{\beta}$ as $\beta \rightarrow 0$. Since $\frac{\alpha}{2} < \beta < \alpha$ as $\alpha \rightarrow 0$ we have that for α sufficiently small

$$f(\beta) \leq f^{1+\delta}(\alpha) .$$

Now we consider the case

$$\lim_{v \rightarrow \infty} \frac{\log a_v}{v} = c > 0 .$$

The proof of theorem (2.2.10) shows that $f(\beta) = O\{f(\alpha)\}$. This completes the proof of the theorem.

Recalling our definition (2.2.2) of the b_μ 's theorem (2.2.11) tells us that

$$\begin{aligned} \sum_{v=0}^{\infty} f_v(\theta) &= \sum_{v=0}^{\infty} \left(\sum_{\mu=1}^{2m-1} \frac{1}{\mu!} (e^{\alpha a_v} - 1)^{-\mu} g_\mu(e^{\alpha a_v}) (i \theta a_v)^\mu \right) + \\ &+ O\left\{ \left(\frac{\theta}{\alpha} \right)^{2m} f^{1+\epsilon}(\alpha) \right\} \end{aligned}$$

where the O -constant is independent of α . Hence

$$\begin{aligned} I_1 &= \int_{-\theta_0}^{\theta_0} \exp \left\{ \sum_{v=0}^{\infty} \left(\sum_{\mu=2}^{2m-1} \frac{1}{\mu!} (e^{\alpha a_v} - 1)^{-\mu} g_\mu(e^{\alpha a_v}) (i \theta a_v)^\mu \right) + \right. \\ &\quad \left. + O\left\{ \left(\frac{\theta}{\alpha} \right)^{2m} f^{1+\epsilon}(\alpha) \right\} \right\} d\theta ; \end{aligned}$$

since

$$\sum_{v=0}^{\infty} (e^{\alpha a_v} - 1)^{-1} g_1(e^{\alpha a_v}) i \theta a_v = i \theta \sum_{v=0}^{\infty} (e^{\alpha a_v} - 1)^{-1} a_v$$

$$= i \theta n ,$$

according to (2.1.18). If we let

$$(2.2.12) \quad A_{\mu} = \sum_{v=0}^{\infty} a_v^{\mu} g_{\mu}(e^{\alpha a_v}) (e^{\alpha a_v} - 1)^{-\mu}$$

we obtain

$$I_1 = \int_{-\theta_0}^{\theta_0} \exp \left\{ -\frac{1}{2} A_2 \theta^2 + \sum_{\mu=3}^{2m-1} \frac{A_{\mu}}{\mu!} (i\theta)^{\mu} + O\left(\left(\frac{\theta}{\alpha}\right)^{2m} f^{1+\epsilon}(\alpha)\right) \right\} d\theta .$$

(2.2.13) Remark: We wish to show that the terms in this series are of decreasing order (with $\theta_0 = \alpha f^{-1/3 - \eta/3}(\alpha)$). This will enable us to expand the integrand in a series of terms of decreasing order and hence obtain an asymptotic expansion of I_1 . Hence we prove

(2.2.14) Theorem: Let A have property P . Then

$$\alpha^{\mu} A_{\mu} = O\{f^{1+\epsilon}(\alpha)\} , \quad \epsilon > 0 , \mu \geq 3 ,$$

where the O -constant may depend upon ϵ and μ but not upon α .

Proof: Since g_{μ} is a polynomial of degree $\leq \mu-1$,

$$\alpha^\mu A_\mu = 0 \left\{ \sum_{\nu=0}^{\infty} (\alpha a_\nu)^\mu e^{(\mu-1)\alpha a_\nu} (e^{\alpha a_\nu} - 1)^{-\mu} \right\}.$$

By arguments completely analogous to those of lemma (2.2.6) we obtain

$$\alpha^\mu A_\mu = 0 \left\{ \sum_{\nu=0}^{\infty} [1 + (\alpha a_\nu)^\mu] e^{-\alpha a_\nu} \right\}.$$

Thus the theorem follows by theorems (2.2.8), (2.2.9) and (2.2.10).

(2.2.15) Remark: Up to this point we have needed only $\theta_0 \leq \alpha f^{-\delta}(\alpha)$ for some fixed $\delta > 0$. To obtain the asymptotic estimate for I_1 however we want

$$(i) \quad \theta_0 A_2^{1/2} \rightarrow \infty \quad \text{as } \alpha \rightarrow 0 \quad (\text{this requires that } \theta_0 > \alpha f^{-1/2}(\alpha))$$

and

$$(ii) \quad A_\mu \theta_0^\mu \rightarrow 0 \quad \text{as } \alpha \rightarrow 0, \quad (\text{which requires that}$$

$$\theta_0 < \alpha f^{-1/3}(\alpha)).$$

Thus we choose $\theta_0 = \alpha f^{-\frac{1}{3} - \frac{\eta}{3}}(\alpha)$, although any number γ , $\frac{1}{2} > \gamma > \frac{1}{3}$ could be used instead of $\frac{1}{3} + \frac{\eta}{3}$.

Now with our choice of θ_0 , by theorem (2.2.14), we have

$$A_{3+j} \theta^{3+j} = O\left(\left(\frac{\theta}{\alpha}\right)^{3+j} f^{1+\epsilon}(\alpha)\right), \quad j = 0, 1, \dots, 2m-3$$

(where the 0-constants are independent of α)

$$= 0 \{f(\alpha)^{1+\epsilon - \frac{(1+\eta)(3+j)}{3}}\}.$$

Thus

$$\exp \left[\sum_{\mu=3}^{2m-1} \frac{A^\mu}{\mu!} (i\theta)^\mu + 0 \{f^{1+\epsilon - \frac{2m(1+\eta)}{3}}(\alpha)\} \right]$$

can be expanded as

$$1 + E(\theta) + 0 \{f^{1 - \frac{2m}{3}}(\alpha)\} \quad (\text{since we can take } \epsilon < \frac{2m}{3} \eta),$$

where

$$E(\theta) = \sum_{v=1}^{5m} \sum_{\mu_1=3}^{2m-1} \cdots \sum_{\mu_v=3}^{2m-1} \frac{1}{v!} \frac{A^{\mu_1} A^{\mu_2} \cdots A^{\mu_v}}{\mu_1! \mu_2! \cdots \mu_v!} (i\theta)^{\mu_1 + \mu_2 + \cdots + \mu_v}.$$

We now proceed as in the paper of Roth and Szekeres [14]. We put

$$\theta = t / \sqrt{\frac{1}{2} A_2}, \quad \text{so that with } B_0 = \theta_0 \sqrt{A_2/2}$$

$$\begin{aligned} I_1 &= \int_{-\theta_0}^{\theta_0} e^{-\frac{A_2 \theta^2}{2}} [1 + E(\theta) + 0 \{f^{1 - \frac{2m}{3}}(\alpha)\}] d\theta \\ &= \left(\frac{1}{2} A_2\right)^{-\frac{1}{2}} \int_{-B_0}^{B_0} e^{-t^2} [1 + E\{t(\frac{1}{2} A_2)^{-\frac{1}{2}}\} + 0 \{f^{1 - \frac{2m}{3}}(\alpha)\}] dt. \end{aligned}$$

Now for each possible ρ, v we group together those terms in the multiple sum defining $E(t(\frac{1}{2} A_2)^{-1/2})$ for which

$$(2.2.16) \quad \mu_1 + \mu_2 + \dots + \mu_v = \sigma \quad \text{and} \quad \rho = \sigma - 2v \quad .$$

In the multiple sum defining E each μ is at least 3 thus (2.2.16) implies that $3v \leq \sigma = \rho + 2v$, thus $v \leq \rho$ and $\sigma \leq 3\rho$. Moreover (2.2.16) implies that $\rho < \sigma < v \leq 10m \leq 10m^2$. Thus

$$E(t \left(\frac{1}{2} A_2\right)^{-\frac{1}{2}}) = \sum_{\rho=1}^{10m^2} \sum_{\sigma=3}^{3\rho} C_{\rho\sigma} (it)^\sigma ,$$

where

$$C_{\rho\sigma} = \sum_{v=1}^{5m} \sum_{\mu_1=3}^{2m-1} \dots \sum_{\mu_v=3}^{2m-1} \frac{1}{v! \mu_1! \dots \mu_v!} A_{\mu_1 \mu_2 \dots \mu_v} \left(\frac{1}{2} A_2\right)^{-\frac{\rho}{2}} ,$$

subject to (2.2.16). Thus

$$I_1 = \left(\frac{1}{2} A_2\right)^{-\frac{1}{2}} \int_{-B_0}^{B_0} e^{-t^2} \left\{ 1 + \sum_{\rho=1}^{2m-1} \sum_{\sigma=3}^{3\rho} C_{\rho\sigma} (it)^\sigma + O\left\{ t^{1 - \frac{3m}{3}(\alpha)} \right\} \right\} dt .$$

Since $B_0 = \theta \sqrt{\frac{1}{2} A_2}$, we now estimate A_2 . It is easily seen that

$$\alpha^2 A_2 = \sum_{v=0}^{\infty} \frac{(\alpha a_v)^2}{e^{\alpha a_v} - 1} + \sum_{v=0}^{\infty} \frac{(\alpha a_v)^2}{(e^{\alpha a_v} - 1)^2} .$$

There exists a number C such that $\alpha a_v > C$ implies that

$$(\alpha a_v)^2 (e^{\alpha a_v} - 1)^{-1} > e^{-\alpha a_v} .$$

Also

$$(\alpha a_v)^2 (e^{\alpha a_v} - 1)^{-2} > K e^{-\alpha a_v}$$

for $\alpha a_v < C$, if K is sufficiently small. Thus $\alpha^2 A_2 > K f(\alpha)$ hence $B_0 > K f^{\eta/3}(\alpha)$. This shows that

$$\begin{aligned} \int_{B_0}^{\infty} e^{-t^2} \left\{ 1 + \sum_{\rho=1}^{2m-1} \sum_{\sigma=3}^{3\rho} C_{\rho\sigma} (it)^{\sigma} + O \left(f^{1 - \frac{2m}{3}} \left(\frac{1}{\alpha} \right) \right) \right\} dt = \\ = O \left\{ \exp \left(- f^{\frac{\eta}{3}}(\alpha) \right) \right\}, \end{aligned}$$

hence, noting also that by symmetry the terms involving odd powers of θ vanish,

$$\begin{aligned} I_1 = \left(\frac{1}{2} A_2 \right)^{-\frac{1}{2}} \left[\int_{-\infty}^{\infty} e^{-t^2} \left\{ 1 + \sum_{\rho'=1}^{m-2} \sum_{\sigma'=3}^{3\rho'} (-1)^{\sigma'} C_{2\rho', 2\sigma'} t^{2\sigma'} \right\} dt \right. \\ \left. + O \left\{ f^{1 - \frac{2m}{3}}(\alpha) \right\} \right]. \end{aligned}$$

The 0-constant here depends upon ϵ and m but not α . We thus conclude that

$$(2.2.14) \quad I_1 = (2\pi/A_2)^{\frac{1}{2}} \left\{ 1 + \sum_{\rho=1}^{m-2} D_{\rho} + O \left(f^{1 - \frac{2m}{3}}(\alpha) \right) \right\},$$

where

$$D_{\rho} = \sum_{\sigma=3}^{3\rho} e_{2\sigma} C_{2\rho, 2\sigma}$$

for certain constants $e_{2\sigma}$. Now

$$C_{2\rho, 2\sigma} = A_2^{-6\rho} 2^\sigma \sum_{v=1}^{5m} \sum_{\mu_1=3}^{2m-1} \cdots \sum_{\mu_v=3}^{2m-1} \frac{A_{\mu_1} A_{\mu_2} \cdots A_{\mu_v}}{\mu_1! \cdots \mu_v!} A_2^{5\rho-v}$$

where $\mu_1 + \mu_2 + \cdots + \mu_v = 2\rho + 2\rho$. Since $2\rho \geq v$ for each term of this multiple sum and writing

$$A_2^{5\rho-v} = \sum_{\mu_{v+1}=2}^2 \cdots \sum_{\mu_{5\rho}=2}^2 A_{\mu_{v+1}} \cdots A_{\mu_{5\rho}},$$

we obtain

$$C_{2\rho, 2\sigma} = 2^\sigma A_2^{-6\rho} \sum_{\mu_1=3}^{2m-1} \cdots \sum_{\mu_v=3}^{2m-1} \sum_{\mu_{v+1}=2}^2 \cdots \sum_{\mu_{5\rho}=2}^2 A_{\mu_1} A_{\mu_2} \cdots A_{\mu_{5\rho}} \sum_{v=1}^{5m}$$

$$\frac{1}{v! \mu_1! \cdots \mu_v!};$$

thus the D 's are of the form in theorem (2.1.13).

Clearly we now need to investigate the integrals I_2 and I_3 of (2.1.19) to complete our proof of theorem (2.1.13). This we shall do in the next section of this chapter.

§2.3 The Estimation of the Integrals I_2 and I_3

First of all since I_3 can be treated analogously to I_2 we shall consider only I_2 .

Throughout this section we let

$$G(\theta) = \prod_{v=0}^{\infty} \frac{1 - e^{-\alpha a_v}}{1 - e^{-\alpha a_v + i a_v \theta}}.$$

We shall break up I_2 (defined by (2.2.9)) into three integrations. Let $1 > \delta > 0$, then

$$\begin{aligned} (2.3.1) \quad I_2 &= \int_{\theta_0}^{\pi} G(\theta) e^{-ni\theta} d\theta = \int_{\theta_0}^{\alpha} + \int_{\alpha}^{\alpha f^{\delta}(\alpha)} + \int_{\alpha f^{\delta}(\alpha)}^{\pi} = \\ &= I_2' + I_2'' + I_2''' . \end{aligned}$$

We shall consider each of these integrals separately. I_2' and I_2'' shall be treated using methods similar to those of Roth and Szekeres in [14].

We note that

$$\begin{aligned} |G(\theta)| &= \left| \exp \left(- \sum_{v=0}^{\infty} \log (1 - e^{-\alpha a_v + i a_v \theta}) (1 - e^{-\alpha a_v})^{-1} \right) \right| \\ &= \exp \left(- \frac{1}{2} \sum_{v=0}^{\infty} \log \left(1 + \frac{2 e^{\alpha a_v} (1 - \cos a_v \theta)}{(e^{\alpha a_v} - 1)^2} \right) \right) . \end{aligned}$$

since $\operatorname{Re} (\log f(z)) = \log |f(z)| = \frac{1}{2} \log |f(z)|^2$. We now prove

(2.3.2) Theorem: Let

$$A\left(\frac{C}{\alpha}\right) > f^{\frac{2}{3} + \eta}(\alpha)$$

where $\frac{1}{3} \geq \eta > 0$, for some C , $\pi \geq C > 0$. Then with fixed $N > 0$

$$\int_{\theta_0}^{\alpha} G(\theta) e^{-ni\theta} d\theta = O\{\alpha f^{-N}(\alpha)\}.$$

The 0-constant depends upon N but not α or n .

Proof: It is easily seen that

$$1 - \cos a_v \theta > \frac{(a_v \theta)^2}{K'}$$

for $a_v \theta < \pi$ if $K' > 0$ is sufficiently large. Also if $\alpha a_v < C$,

$$\frac{2 e^{\alpha a_v}}{(e^{\alpha a_v} - 1)^2} > \frac{1}{K''(\alpha a_v)^2}.$$

Thus $a_v \theta \leq \alpha a_v < C$ implies that

$$|G(\theta)| < \exp \left\{ -\frac{k}{2} \log \left(1 + \frac{K'' \theta^2}{K' 3\alpha^2} \right) \right\}$$

where k is the number of a_v such that

$\alpha a_v \leq C$, i.e. $k = A(\frac{C}{\alpha})$. This is

$$\begin{aligned} &< \exp \left\{ -\frac{k}{2} \log \left(1 + \frac{\theta^2}{K\alpha} \right) \right\} , \quad K = \frac{K'}{K''} \\ &\leq \exp \left\{ -\frac{k}{2K} f^{-\frac{2}{3} - \frac{2\eta}{3}}(\alpha) \right\} \end{aligned}$$

(using the value of θ_0 given in (2.1.5)).

$$\leq \exp \left\{ -A\left(\frac{C}{\alpha}\right) f^{-\frac{2}{3} - \frac{2\eta}{3}}(\alpha) \right\} .$$

Since $A(\frac{C}{\alpha}) > f^{\frac{2}{3} + \eta}(\alpha)$,

$$|G(\theta)| < \exp \left\{ -f^{\frac{\eta}{3}}(\alpha) \right\} = O\{f^{-N}(\alpha)\}$$

for each $N > 0$ and this completes the proof of the theorem.

Next we consider I_2'' .

(2.3.3) Theorem: Suppose that for some δ , $0 < \delta < 1$, and some C ,
 $\pi \geq C > 0$

$$A\left(\frac{C f^{-\delta}(\alpha)}{\alpha}\right) / \log f(\alpha) \rightarrow \infty$$

as $\alpha \rightarrow 0$. Let N be any fixed integer > 0 . Let $n > 0$ be arbitrary.

Then

$$I_2'' = \int_{\alpha}^{\alpha f^{\delta}(\alpha)} G(\theta) e^{-in\theta} d\theta = O\{\alpha f^{-N}(\alpha)\} \quad (\alpha \rightarrow 0) \quad .$$

The constant in the 0-term depends upon δ and N but not α or n .

Proof: The proof is similar to that of theorem (2.3.2). We take here $\theta a_v \leq C$ for $v = 1, 2, \dots, k$. Thus as before

$$\begin{aligned} |G(\theta)| &\leq \exp \left\{ -\frac{k}{2K} \left(\frac{\theta}{\alpha}\right)^2 \right\} \\ &\leq \exp \left\{ -\frac{1}{2K} A\left(\frac{C}{\alpha f^{\delta}(\alpha)}\right) \right\} \quad . \end{aligned}$$

Since $A\left(\frac{C}{\alpha f^{\delta}(\alpha)}\right) / \log f(\alpha) \rightarrow \infty$ we are done.

There is still I_2''' to consider. Recall that

$$G(\theta) = \prod_{v=0}^{\infty} \left(\frac{1 - e^{-\alpha a_v}}{1 - e^{-\alpha a_v + i a_v \theta}} \right) \quad .$$

We shall consider each of the terms in this product separately. For this purpose we prove:

(2.3.4) Lemma: Let ϕ be any fixed number with $0 < \phi < \pi$. If

$\theta \in [\phi, 2\pi - \phi]$ then

$$\left| \frac{1 - e^{-\alpha a_v}}{1 - e^{-\alpha a_v + i\theta}} \right| \leq \frac{1}{(1 - \cos \phi)^{1/2}} \cdot \frac{1 - e^{-\alpha a_v}}{(1 + e^{-2\alpha a_v})^{1/2}} \quad .$$

Proof: Suppose $\theta \in [\phi, \pi]$.

If we construct the triangle with vertices at $0, 1, e^{-\alpha a_v + i\theta}$,
we see that by the cosine law for triangles

$$\begin{aligned} \left| 1 - e^{-\alpha a_v + i\theta} \right| &= (1 + e^{-2\alpha a_v} - 2 e^{-\alpha a_v} \cos \theta)^{1/2} \\ &= (1 + e^{-2\alpha a_v})^{1/2} \left(1 - \frac{2 e^{-\alpha a_v}}{1 + e^{-2\alpha a_v}} \cos \theta \right)^{1/2} . \end{aligned}$$

Since $0 < \frac{2 e^{-\alpha a_v}}{1 + e^{-2\alpha a_v}} \leq 1$, this yields that

$$\begin{aligned} \left| 1 - e^{-\alpha a_v + i\theta} \right| &\geq (1 + e^{-2\alpha a_v})^{1/2} (1 - \cos \theta)^{1/2} , \\ &\geq (1 + e^{-2\alpha a_v})^{1/2} (1 - \cos \phi)^{1/2} . \end{aligned}$$

The case $\theta \in [\pi, 2\pi - \phi]$ is treated similarly.

(2.3.5) Theorem: Let A be a Q -sequence (definition (2.1.2)). Let ϵ be an arbitrary constant > 0 . Let n be an arbitrary integer. Let $N > 0$ be an arbitrary fixed integer. Then

$$\int_{\alpha \frac{\epsilon}{N}}^{\pi} G(\theta) e^{-ni\theta} d\theta = O(\alpha^{N-\epsilon}) .$$

The O -constant may depend upon N and ϵ but not upon n or α .

Proof: For each integer a_v we construct the sets

$$\begin{aligned} \Delta_{(v)}^1 &= \left[\frac{2\pi - \alpha \frac{\epsilon}{N}}{a_v}, \frac{2\pi + \alpha \frac{\epsilon}{N}}{a_v} \right] \\ &\vdots \\ \Delta_{(v)}^j &= \left[\frac{j 2\pi - \alpha \frac{\epsilon}{N}}{a_v}, \frac{j 2\pi + \alpha \frac{\epsilon}{N}}{a_v} \right] \end{aligned}$$

where $j \leq [a_v/2]$.

We note that all $\theta \in \Delta_{(\ell)}^j$ are mapped into the interval

$$[j 2\pi - \alpha \frac{\epsilon}{N}, j 2\pi + \alpha \frac{\epsilon}{N}]$$

under the map $\theta \rightarrow \theta a_v$. Hence all $\theta \in \bigcup_{j=1}^{[a_v/2]} \Delta_{(v)}^j$ are mapped into the interval $[-\alpha \frac{\epsilon}{N}, \alpha \frac{\epsilon}{N}] \bmod 2\pi$ under the map $\theta \rightarrow \theta a_v$. Moreover these are the only $\theta \in [0, \pi]$ which are mapped into $[-\alpha \frac{\epsilon}{N}, \alpha \frac{\epsilon}{N}] \bmod 2\pi$ under the map $\theta \rightarrow a_v \theta$.

Let $a_\ell > 1$ be some element of A and let (m, n) denote the greatest common divisor of the integers m and n . Suppose for some j the equation

$$\frac{j}{a_\ell} = \frac{k}{a_v}$$

is solvable for each $a_v \in A$ with $v > \ell$. This implies that

$$\frac{a_\ell}{(a_{\ell,j})} \mid a_v$$

for each $v > \ell$. Since $j \leq \frac{a_\ell}{2}$, we have that $a_\ell / (a_{\ell,j}) \geq 2$ and thus we are lead to a contradiction since A is a Q-sequence. Thus for each j there exists an $a_\ell(j) \in A$ such that

$$\frac{j \cdot 2\pi}{a_\ell} = \frac{k \cdot 2\pi}{a_\ell(j)}$$

is not solvable.

If $\theta \notin \bigcup_{j=1}^{\lfloor \frac{a_\ell}{2} \rfloor} \Delta_{(\ell)}^j$, then from lemma (2.3.4) we obtain that

$$\frac{1 - e^{-\alpha a_\ell}}{1 - e^{-\alpha a_\ell + i a_\ell \theta}} = O(\alpha) \quad O(\alpha^{-\frac{\epsilon}{N}}) = O(\alpha^{1 - \frac{\epsilon}{N}}),$$

as $\alpha \rightarrow 0$.

For sufficiently small α

$$\Delta_{(\ell)}^j \cap \bigcup_{k=1}^{\lfloor a_\ell(j)/2 \rfloor} \Delta_{(a_\ell(j))}^k = \emptyset.$$

Thus if $\theta \in \Delta_{(\ell)}^j$ from lemma (2.3.4) we obtain that

$$\frac{1 - e^{-\alpha a_{\ell}(j)}}{1 - e^{-\alpha a_{\ell}(j) + i a_{\ell}(j)\theta}} = O(\alpha^{1 - \frac{\epsilon}{N}}).$$

We thus conclude that the product

$$\prod_j \frac{1 - e^{-\alpha a_{\ell}}}{1 - e^{-\alpha a_{\ell} + i a_{\ell}\theta}} = O(\alpha^{1 - \frac{\epsilon}{N}})$$

where $\prod_j$ means that the product is taken over the distinct $a_{\ell}(j)$.

Let a'_{ℓ} be the smallest element of A which is greater than each of the $a_{\ell}(j)$. Then we may repeat the above argument with a'_{ℓ} replacing a_{ℓ} . Since we may do this N -times we obtain that

$$G(\theta) = O(\alpha^{N-\epsilon}) ;$$

from which the theorem follows at once.

(2.3.6) Theorem: Let N and n be arbitrary fixed integers > 0 .

Let

$$\frac{1}{\alpha^4} = O\{f^{-N}(\alpha)\}$$

for all $\alpha \in A$, where A is a subset of the positive reals and

$$\min_{\alpha \in A} \alpha = 0.$$

Then

$$\int_{\alpha \frac{1}{4}}^{\pi} G(\theta) e^{-ni\theta} d\theta = O\{\alpha f^{-N}(\alpha)\}$$

as $\alpha \rightarrow 0$ through elements of A . The O -constant depends upon N and ϕ but not n or α .

Proof: Since we are not considering sets A for which every element of A is divisible by the same number we obtain as in the proof of theorem (2.3.5) with $a_\ell = a_0$, $N = 1$, $\epsilon = \frac{1}{4}$ that there is a subproduct of $G(\theta)$ which is $O(\alpha^{3/4})$. If $a_0 = 1$, this follows immediately from lemma (2.3.4).

Now consider some other term,

$$\frac{1 - e^{-\alpha a_v}}{1 - e^{-\alpha a_v + i a_v \theta}}$$

where v is sufficiently large and fixed. Then as before we construct the sets

$$\Delta_{(v)}^j = \left[\frac{j 2\pi - \alpha^{1/2}}{a_v}, \frac{j 2\pi + \alpha^{1/2}}{a_v} \right].$$

If $\theta \notin \Delta_{(v)}^j$ then $(1 - \cos a_v \theta)^{-1/2} \leq \alpha^{-1/2}$. Thus

$$\frac{1 - e^{-\alpha a_v}}{1 - e^{-\alpha a_v + i a_v \theta}} = O(\alpha^{1/2}) \quad \text{as } \alpha \rightarrow 0,$$

if $\theta \notin \Delta_{(v)}^j$ for some j . However those $\theta \in \bigcup_j \Delta_{(v)}^j$ form a set of measure M , where

$$M \leq \frac{a_v}{2} \times \frac{2 \alpha^{\frac{1}{2}}}{a_v} = \alpha^{\frac{1}{2}}.$$

Let

$$G_1(\theta) = \prod_{l \geq v} \frac{1 - e^{-\alpha a_l}}{1 - e^{-\alpha a_l} + i a_l \theta}.$$

Then

$$\begin{aligned} \int_{\phi}^{\pi} |G(\theta)| |d\theta| &= O(\alpha^{\frac{3}{4}} \int_{\alpha^{\frac{1}{4}}}^{\pi} |G_1(\theta)| d\theta) = \\ &= O\left\{ \alpha^{\frac{3}{4}} \int_{\theta \notin \bigcup_j \Delta_{(v)}^j} |G_1(\theta)| d\theta + \alpha^{\frac{3}{4}} \int_{\theta \in \bigcup_j \Delta_{(v)}^j} |G_1(\theta)| d\theta \right\} \\ &= O\left\{ (2\alpha^{\frac{3}{4}} \alpha^{\frac{1}{2}}) = O(\alpha^{1 + \frac{1}{4}}) \right. \\ &= O(\alpha f^{-N}(\alpha)) \quad , \quad N > 0 \quad , \end{aligned}$$

since $f^{-N}(\alpha) = O(\alpha^{1/4})$ for $\alpha \in A$. This completes the proof.

(2.3.7) Theorem: Let N, n be an arbitrary fixed integers > 0 . Let δ be an arbitrary constant with $0 < \delta < 1$. Define

$$M = \left[\frac{N}{\delta} \right] + 1 .$$

Then

$$\int_{\alpha f^{\delta}(\alpha)}^{\frac{3\pi}{2a_M}} G(\theta) e^{-in\theta} d\theta = O\{f^{-N}(\alpha)\} .$$

The 0-constant depends upon N and δ but not α or n .

Proof: Since $a_v \theta \leq \frac{3\pi}{2}$ for $v = 1, 2, \dots, M$;

$$a_v \theta \in [\alpha f^{\delta}(\alpha) , \frac{3\pi}{2}]$$

for these v . Since

$$(1 - \cos \alpha f^{\delta}(\alpha))^{-\frac{1}{2}} = O\{\alpha^{-1} f^{-\delta}(\alpha)\}$$

as $\alpha \rightarrow 0$, if $v \leq M$, by lemma (2.3.4) with $\theta = \alpha f^{\delta}(\alpha)$

$$\frac{1 - e^{-\alpha a_v}}{1 - e^{-\alpha a_v + i a_v \theta}} = O(\alpha^{-1} f^{-\delta}(\alpha)) O(\alpha) = O(f^{-\delta}(\alpha)) .$$

$$\text{Thus } |G(\theta)| \leq \prod_{v \leq M} |1 - e^{-\alpha a_v}| |1 - e^{-\alpha a_v + i a_v \theta}|^{-1}$$

$$= O\{ \prod_{v \leq M} f^{-\delta}(\alpha) \}$$

$$= O\{ f^{-\delta(\left[\frac{N}{\delta} \right] + 1}(\alpha) \}$$

$$= O\{f^{-N}(\alpha)\} \quad .$$

This proves theorem (2.3.7).

(2.3.8) Theorem: Let N and n be arbitrary fixed integers > 0 .

Let ψ be a fixed number > 0 and suppose that

$$f(\alpha) = O(\alpha^{-\psi}) \quad ,$$

for all $\alpha \in \mathcal{B}$, where $\mathcal{B}$ is a subset of the positive reals and

$$\min_{\alpha \in \mathcal{B}} \alpha = 0 \quad .$$

Let δ be any fixed constant $0 < \delta < \frac{1}{10}$. Define M to be $\max \left(\left[\frac{N}{8} \right] + 1, 8(1+N\psi) \right)$. Then

$$\int_{\alpha f^{\delta}(\alpha)}^{\frac{3}{2}a_M} G(\theta) e^{-in\theta} d\theta = O\{\alpha f^{-N}(\alpha)\}$$

as $\alpha \rightarrow 0$ through elements of M . The O -constant depends upon N and δ but not α or n .

Proof: Let

$$J_1 = \int_{\alpha \frac{7}{8}}^{\frac{3\pi}{2}a_M} G(\theta) e^{-in\theta} d\theta \quad .$$

Since $a_v \theta \leq \frac{3\pi}{2}$ for $v = 1, 2, \dots, M$ we obtain that $J_1 = O(\alpha^{\frac{M}{8}})$ by the proof of theorem (2.3.7) with $f^\delta(\alpha)$ replaced by $\alpha^{-1/8}$. Since

$$\frac{M}{8} > 1 + N \psi ,$$

we obtain that

$$J_1 = O\{\alpha f^{-N}(\alpha)\} .$$

We now consider

$$J_2 = \int_{\alpha f^\delta(\alpha)}^{\alpha \frac{7}{8}} G(\theta) e^{-in\theta} d\theta .$$

As in the proof of theorem (2.3.7) we obtain that the product of the first M terms of $G(\theta)$ is $O\{f^{-N}(\alpha)\}$. Let a_K be the smallest element of A such that $a_K > \alpha^{-1} f^{-\delta}(\alpha)$. Let us again define

$$\Delta_{(K)}^j = \left[\frac{j 2\pi - \alpha^{1/2}}{a_K} , \frac{j 2\pi + \alpha^{1/2}}{a_K} \right]$$

for $j = 1, 2, \dots, [\frac{a_K}{2}]$. Each $\theta \in \Delta_{(K)}^j$ is mapped into $[j 2\pi - \alpha^{1/2} , j 2\pi + \alpha^{1/2}]$ under the map $\theta \rightarrow \theta a_K$. For every $\theta \in (\alpha f^\delta(\alpha) , \pi)$ which is not an element of

$$\bigcup_{j=1}^{[a_K/2]} \Delta_{(K)}^j , \quad a_K \theta \in [\alpha^{1/2} , 2\pi - \alpha^{1/2}] \quad \text{and for these } \theta$$

$$\frac{1 - e^{-\alpha a_K}}{1 - e^{-\alpha a_K + i a_K \theta}} = O(\alpha^{\frac{1}{2}})$$

by lemma (2.3.4). Let

$$M_1 = [\alpha f^\delta(\alpha), \alpha^{\frac{7}{8}}] \cap \bigcup_{j=1}^{[a_K/2]} \Delta_{(K)}^j,$$

and M_2 be the complement of M_1 , relative to $[\alpha f^\delta(\alpha), \alpha^{7/8}]$. Then

$$\int_{M_2} |G(\theta)| d\theta = O\{\alpha^{1/2} f^{-N}(\alpha) \int_{M_2} 1 \cdot d\theta\}.$$

Since the measure of M_2 is less than $\alpha^{7/8}$ we obtain that

$$\int_{M_2} |G(\theta)| d\theta = O\{f^{-N}(\alpha) \alpha^{11/8}\} = O\{\alpha f^{-N}(\alpha)\}.$$

Let us now consider

$$\int_{M_1} |G(\theta)| d\theta = O\{f^{-N}(\alpha) \int_{M_1} 1 \cdot d\theta\}.$$

Let $[\alpha f^\delta(\alpha), \alpha^{7/8}]$ be divided into equal lengths of $\frac{2\pi}{a_K}$.

We may assume there are at least two. The number of $\Delta_{(K)}^j \subset [\alpha f^\delta(\alpha), \alpha^{7/8}]$ does not exceed the number of such lengths by more than 2. The length of each $\Delta_{(K)}^j$ contained in $[\alpha f^\delta(\alpha), \alpha^{7/8}]$ is $\leq \frac{\alpha^{1/2}}{\pi} \times \frac{2\pi}{a_K}$. Thus the sum of the lengths of the $\Delta_{(K)}^j$ contained in $[\alpha f^\delta(\alpha), \alpha^{7/8}]$ is

$\leq \frac{\alpha^{1/2}}{\pi} \cdot (\text{the total length of the lengths } \frac{2\pi}{a_K} \text{ contained in } [\alpha f^\delta(\alpha), \alpha^{7/8}]) + 4 \alpha^{1/2} / a_K$. This means that the measure of M_1 is $\leq \alpha^{1/2} \times (\text{measure of } [\alpha f^\delta(\alpha), \alpha^{7/8}]) + 4 \alpha^{3/2} f^\delta(\alpha)$.

Since $f(\alpha) = O(\alpha^{-1})$ and $\delta < \frac{1}{2}$ we see that $4 \alpha^{3/2} f^\delta(\alpha) = O(\alpha)$. Furthermore since the measure of $[\alpha f^\delta(\alpha), \alpha^{7/8}]$ is $O(\alpha^{7/8})$ we conclude that

$$J_2 = O\{\alpha f^{-N}(\alpha)\}.$$

Adding J_1 and J_2 we obtain the theorem.

We may now prove our final result in the estimation of I_2 (hence I_3).

(2.3.9) Theorem: Suppose that for some constant δ with $1 > \delta > 0$ that

$$A\left(\frac{\pi}{\alpha f^\delta(\alpha)}\right) / \log f(\alpha) \rightarrow \infty$$

as $\alpha \rightarrow 0$. Suppose furthermore that

$$A\left(\frac{\pi}{\alpha}\right) > f^{\frac{2}{3} + \eta}(\alpha)$$

where η is some constant with $\frac{1}{3} \geq \eta > 0$. Let $\epsilon > 0$ be an arbitrary constant. Choose N to be any fixed integer > 0 .

Let A satisfy either of (i) or (ii) below:

$$(i) \quad \lim_{\alpha \rightarrow 0} \frac{\log f(\alpha)}{\log \frac{1}{\alpha}} = 0 ;$$

(ii) A is a Q -sequence.

Then (with θ_0 defined by (2.1.20))

$$\int_{\theta_0}^{\pi} G(\theta) e^{-i\eta\theta} d\theta = O\{\alpha f^{-N+\epsilon}(\alpha)\} .$$

The O -constant is independent of α but does depend upon N and ϵ .

Proof: Since

$$A\left(\frac{\pi}{\alpha}\right) > f^{\frac{2}{3} + \eta}(\alpha)$$

we conclude from theorem (2.3.2) that

$$\int_{\theta_0}^{\alpha} G(\theta) e^{-i\eta\theta} d\theta = O\{\alpha f^{-N}(\alpha)\} .$$

Since

$$A\left(\frac{\pi}{\alpha f^{\delta}(\alpha)}\right) / \log f(\alpha) \rightarrow \infty$$

as $\alpha \rightarrow 0$; we conclude from theorem (2.3.2) that

$$\int_{\alpha}^{\alpha f^{\delta}(\alpha)} G(\theta) e^{-in\theta} d\theta = O\{\alpha f^{-N}(\alpha)\} ,$$

where the 0-constant is independent of α and n .

If

$$\lim_{\alpha \rightarrow 0} \frac{\log f(\alpha)}{\log \frac{1}{\alpha}} = 0$$

then in theorem (2.3.8) we may take $\psi = 1/N$ and B to be the positive reals. Since

we are allowed to fix δ as small as necessary, let us fix δ so small that $[\frac{N}{\delta}] + 1 > 8$. ($\delta \leq \frac{1}{8}$) . Then with $M_1 = [\frac{N}{\delta}] + 1$, by theorem (2.3.8)

$$\int_{\alpha f^{\delta}(\alpha)}^{\frac{3\pi}{2a_{M_1}}} G(\theta) e^{-in\theta} d\theta = O\{\alpha f^{-N}(\alpha)\} ,$$

where the 0-constant is independent of α and n .

Moreover in case (i) the set A in theorem (2.3.6) may be chosen to be the positive reals. For sufficiently small α , $\alpha^{1/4} < \frac{3\pi}{2a_{M_1}}$ and since we are taking A to be a 1-sequence; we conclude from theorem (2.3.6) that

$$\int_{\frac{3\pi}{2a_{M_1}}}^{\pi} G(\theta) e^{-in\theta} d\theta = O\{\alpha f^{-N}(\alpha)\theta\} ,$$

where the 0-constant is independent of α and n .

This proves theorem (2.3.9) in case (i).

Let us now consider case (ii). Let $\mathcal{B}$ be those α for which $\log f(\alpha) < \frac{1}{N} \log \frac{1}{\alpha}$. If $\min_{\alpha \in \mathcal{B}} \alpha = 0$ we apply theorem (2.3.8). Again

we fix δ so small that $[\frac{N}{\delta}] + 1 > 8$. Set $M_2 = [\frac{2N}{\delta}] + 1$. Then by theorem (2.3.8)

$$\int_{\alpha f^\delta(\alpha)}^{\frac{3\pi}{2a_{M_2}}} G(\theta) e^{-in\theta} d\theta = O\{\alpha f^{-N}(\alpha)\}$$

as $\alpha \rightarrow 0$ through elements of $\mathcal{B}$. For those α that are not elements of $\mathcal{B}$ we apply theorem (2.3.7). If $\alpha \notin \mathcal{B}$, we have $f(\alpha) \geq \alpha^{-1/N}$.

With $M_2 = [\frac{2N}{\delta}] + 1$ we have by theorem (2.3.7)

$$\int_{\alpha f^\delta(\alpha)}^{\frac{3\pi}{2a_{M_2}}} G(\theta) e^{-in\theta} d\theta = O\{\alpha f^{-N}(\alpha)\}.$$

Thus in case (ii) we conclude that

$$\int_{\alpha f^\delta(\alpha)}^{\frac{3\pi}{2a_{M_2}}} = O\{\alpha f^{-N}(\alpha)\}.$$

Since in case (ii) we assumed that A is a Q -sequence and since

$\alpha = O\{f^{-1}(\alpha)\}$; from theorem (2.3.5) it follows that, for $\alpha^{\epsilon/N} < 3\pi/2 A_{n_2}$

$$\int_{\frac{3\pi}{2a_{M_2}}}^{\pi} G(\theta) e^{-i\eta\theta} d\theta = O \{ \alpha f^{-N+\epsilon}(\alpha) \} .$$

This proves the theorem in case (ii).

We may now complete the proof of theorem (2.1.13). Recall from (2.2.12) that

$$\begin{aligned} A_2 &= \sum_{v=0}^{\infty} a_v^2 g_2(e^{\alpha a_v}) (e^{\alpha a_v} - 1)^{-2} \\ &= \sum_{v=0}^{\infty} a_v^2 e^{\alpha a_v} (e^{\alpha a_v} - 1)^{-2} \\ &= \sum_{v=0}^{\infty} \frac{(a_v)^2}{e^{\alpha a_v} - 1} + \sum_{v=0}^{\infty} \frac{(a_v)^2}{(e^{\alpha a_v} - 1)^2} . \end{aligned}$$

If A has property $\mathcal{P}$, then it follows from remark (2.2.11) that

$$A_2 = O \{ \alpha^{-2} f^{1+\epsilon}(\alpha) \} .$$

Thus $\alpha(A_2)^{1/2} = O \{ f^{1/2 + \epsilon/2}(\alpha) \}$, and theorem (2.1.13) follows from theorem (2.3.9) and equation (2.2.20).

§2.4 Further Asymptotic Relations for Partitions.

In this section we investigate alternate conditions for the asymptotic expansion of theorem (2.1.13) to be valid.

(2.4.1) Remark: Let us consider those A 's for which

$$S = \lim_{v \rightarrow \infty} \frac{\log a_v}{\log v} \quad \text{exists.}$$

For each $\delta > 0$, there exists a $v_0 = v_0(\delta)$ such that for $v \geq v_0$

$$v^{S-\delta} \leq a_v + v^{S+\delta}.$$

It follows at once that $\log 1/\alpha / (S+\delta) < \log A(\frac{1}{\alpha}) < \log \frac{1}{\alpha} / (S-\delta)$ as $\alpha \rightarrow 0$. Thus

(i) $\log A(\frac{1}{\alpha}) \sim \frac{1}{S} \log \frac{1}{\alpha}$, as $\alpha \rightarrow 0$. Moreover, for each fixed η with $0 < \eta < 1$

$$(ii) \quad \log A(\alpha^{\eta-1}) \sim \frac{1-\eta}{S} \log \frac{1}{\alpha} \quad \text{as } \alpha \rightarrow 0.$$

It readily follows by arguments analogous to those in the proof of theorem (2.2.9) that

(iii) $\log f(\alpha) \sim \frac{1}{S} \log \frac{1}{\alpha}$ as $\alpha \rightarrow 0$. From (i) and (ii) we obtain at once that

$$A(\frac{\pi}{\alpha}) > f^{\frac{5}{6}}(\alpha)$$

as $\alpha \rightarrow 0$. From (iii) it follows that $\log f(\alpha) > \log \frac{1}{\alpha} / 2S$ as $\alpha \rightarrow 0$. Hence there exists a fixed δ such that $\delta \log f(\alpha) < \eta \log \frac{1}{\alpha}$. Then

$$\log A\left(\frac{\pi}{\alpha f^{\delta}(\alpha)}\right) > \frac{1-\eta}{2S} \log \frac{1}{\alpha} ;$$

from which it follows that

$$A\left(\frac{\pi}{\alpha f^{\delta}(\alpha)}\right) / \log f(\alpha) \rightarrow \infty$$

as $\alpha \rightarrow 0$. We thus conclude that those A 's for which

$$S = \lim_{v \rightarrow \infty} \frac{\log a_v}{\log v} \quad \text{exists}$$

satisfy the hypothesis of theorem (2.1.13), hence theorem (2.1.13) relaxes condition (i) of the theorem of Roth and Szekeres. Note moreover that we may obtain asymptotic expansions for $P_A(n)$ without considering the condition

$$J_k = \inf_{\frac{1}{2} a_k^{-1} \leq \alpha \leq \frac{1}{2}} \{(\log k)^{-1} \sum_{v=1}^k \left(\frac{a_k}{a_v}\right)^2 ||a_v \alpha||^2\} \rightarrow \infty$$

as $k \rightarrow \infty$. It often requires the theory of trigonometric sums to see when this condition is satisfied (see Roth and Szekeres [14]). Moreover if A is not a Q -sequence we obtain that

$$J'_k = \inf_{\frac{1}{2} a_k^{-1} \leq \alpha \leq \frac{1}{2}} \{ (\log k)^{-1} \sum_{v=1}^k ||a_v \alpha||^2 \} \rightarrow 0$$

as $k \rightarrow \infty$. (If q/a_v for $v \geq v_0$, set $\alpha = \frac{1}{q}$). Thus we conclude that the condition that A is a Q -sequence is less restrictive than the restriction under which Roth and Szekeres obtained asymptotic expansions for partitions into distinct summands.

We now study those A 's for which

$$A(2u) \leq C A(u)$$

for $u \geq u_0$. This condition was used by Schwartz in his papers [7], [12].

(2.4.2) Lemma: If $A(2u) \leq C A(u)$, $u > u_0$ then

$$A\left(\frac{1}{\alpha}\right) > C_0 f(\alpha)$$

Proof: For $a_v \leq \frac{1}{\alpha}$, $1 \geq e^{-\alpha a_v} \leq e^{-1}$, and

$$\sum_{\alpha_v \leq \frac{1}{\alpha}} e^{-\alpha a_v} < A\left(\frac{1}{\alpha}\right)$$

Moreover

$$\begin{aligned} & \begin{matrix} a_v \leq \frac{2}{\alpha} \\ \sum \\ a_v > \frac{1}{\alpha} \end{matrix} e^{-\alpha a_v} < e^{-1} \left[A\left(\frac{2}{\alpha}\right) - A\left(\frac{1}{\alpha}\right) \right] < e^{-1} C A\left(\frac{1}{\alpha}\right), \end{aligned}$$

and in general

$$\begin{aligned} a_v &\leq \frac{2^{j+1}}{\alpha} \\ \sum_{a_v > \frac{2^j}{\alpha}} e^{-\alpha a_v} &< e^{-2^j} \left[A\left(\frac{2^{j+1}}{\alpha}\right) - A\left(\frac{2^j}{\alpha}\right) \right] \leq e^{-2^j} C^{j+1} A\left(\frac{1}{\alpha}\right), \end{aligned}$$

since we may assume $u_0 < \alpha^{-1}$ ($\alpha \rightarrow 0$).

Thus

$$\sum_{v=0}^{\infty} e^{-\alpha a_v} < C A\left(\frac{1}{\alpha}\right) \sum_{j=0}^{\infty} e^{-2^j} C^j.$$

Since

$$\sum_{j=0}^{\infty} e^{-2^j} C^j < \infty$$

we may conclude that

$$f(\alpha) = \sum_{v=0}^{\infty} e^{-\alpha a_v} < K A\left(\frac{1}{\alpha}\right),$$

where $K = C \sum_{j=0}^{\infty} C^j \exp(-2^j b)$. This proves the lemma with $C_0 = K^{-1}$.

(2.4.3) Lemma: Let $A(2u) < C A(u)$, for $u > u_0$. Then there exists

a δ with $0 < \delta < 1$ such that

$$A(\alpha^{-1} f^{-\delta}(\alpha)) / \log f(\alpha) \rightarrow \infty$$

as $\alpha \rightarrow 0$.

Proof: Since $A(2u) < C A(u)$ for $u > u_0$, if $f^{-\delta}(\alpha) \alpha^{-1} > u_0$;

$$A\left(\frac{\alpha^{-1}}{2}\right) > C^{-1} A(\alpha^{-1})$$

and more generally

$$A(\alpha^{-1} 2^{-j}) \geq C^{-j} A(\alpha^{-1}) .$$

Thus

$$A(\alpha^{-1} f^{-\delta}(\alpha)) \geq A(\alpha^{-1} 2^{-(1+[\delta \log_2 f(\alpha)])}) ,$$

since

$$\begin{aligned} A(\alpha^{-1} 2^{-(1+[\delta \log_2 f(\alpha)])}) &\geq C^{-(1+[\delta \log_2 f(\alpha)])} A(\alpha^{-1}) \\ &\geq C^{-(1+\delta \log_2 f(\alpha))} A(\alpha^{-1}) \\ &\geq C^{-1} f^{-\delta \log_2 C}(\alpha) A(\alpha^{-1}) , \end{aligned}$$

from lemma (2.4.2) it follows, if δ is chosen small enough so that

$\delta \log_2 C \leq \frac{1}{2}$, that

$$A(\alpha^{-1} f^{-\delta}(\alpha)) \geq C_1 f^{\frac{1}{2}}(\alpha) , \quad C_1 > 0 .$$

This completes the proof of the lemma since $f^{\frac{1}{2}}(\alpha) / \log f(\alpha) \rightarrow \infty$ as $\alpha \rightarrow 0$.

(2.4.4) Lemma: Let $A(2u) \leq C A(u)$ for $u \geq u_0$. Then as in theorem (2.2.14)

$$\sum_{v=0}^{\infty} f_v(\theta) = \sum_{v=0}^{\infty} \sum_{\mu=0}^{2m-1} \frac{b_{\mu}}{\mu!} (i \theta a_v)^{\mu} + O \left\{ \left(\frac{\theta}{\alpha} \right)^{2m} f(\alpha) \right\} ,$$

where the O -constant depends upon m but not upon α .

Proof: The proof will follow as in theorem (2.2.14) if we can prove the following two facts:

$$(1) \quad \sum_{v=0}^{\infty} (\alpha a_v)^{\mu} e^{-\alpha a_v} = O\{f(\alpha)\}$$

$$(2) \quad f(\beta) = O\{f(\alpha)\} \quad \text{where} \quad \beta = \alpha \left(1 - f^{-\frac{1}{3}} - \frac{\eta}{3}(\alpha) \right) .$$

Statement (1) can be proved by showing that

$$\sum_{v=0}^{\infty} (\alpha a_v)^{\mu} e^{-\alpha a_v} = O\left\{A\left(\frac{1}{\alpha}\right)\right\} .$$

This can be proven in the same way that lemma (2.4.2) was proven.

Since $f(\beta) = 0 \{A(\frac{1}{\alpha})\}$ (lemma (2.4.2) and since $\beta^{-1} \leq 2\alpha^{-1}$ we easily obtain statement (2), thereby completing the proof of lemma (2.4.4).

We now prove

(2.4.5) Theorem: Suppose, for some constant C ,

(2.4.6) $A(2u) < C A(u)$ for $u \geq u_0$. Suppose either that

$$\lim_{\alpha \rightarrow 0} \frac{\lim_{\alpha \rightarrow 0} f(\alpha)}{\log \frac{1}{\alpha}} \rightarrow 0$$

or that A is a Q-sequence. Then the asymptotic relation for $P_A(n)$ given in theorem (2.1.13) holds.

Proof: Replacing theorem (2.2.11) by lemma (2.4.4) yields that theorem (2.1.13) may be deduced with condition (2.4.6) replacing property P . Lemmas (2.4.2) and (2.4.3) replace the conditions

$$A(\frac{\pi}{\alpha}) > f^{\frac{2}{3} + \eta}(\alpha), \quad \eta > 0 \text{ and fixed,}$$

and

$$A(\frac{\pi}{\alpha f^{\delta}(\alpha)}) / \log f(\alpha) \rightarrow \infty$$

respectively. Thus the theorem follows from theorem (2.1.13).

(2.4.7) Remark: Theorem (2.4.5) gives more general results for $P_A(n)$ than does theorem 5 of Schwarz in [12]. Moreover the error terms may be investigated further.

(2.4.8) Theorem: If

$$\lim_{v \rightarrow \infty} \frac{\log a_v}{v} > 0$$

then, with $C > 0$ some constant,

$$A\left(\frac{\pi}{\alpha}\right) > C f(\alpha) .$$

Proof: Let $\lim_{v \rightarrow \infty} \frac{\log a_v}{v} = C$. There exists a v_0 such that $v \geq v_0$ implies that $a_v > e^{Cv/2}$. Now

$$f(\alpha) = \sum_{a_v < \frac{\pi}{\alpha}} e^{-\alpha a_v} + \sum_{a_v \geq \frac{\pi}{\alpha}} e^{-\alpha a_v} .$$

Certainly

$$\sum_{a_v < \frac{\pi}{\alpha}} e^{-\alpha a_v} < A\left(\frac{\pi}{\alpha}\right) .$$

Moreover

$$\begin{aligned}
 \sum_{a_v \geq \frac{\pi}{\alpha}} e^{-\alpha a_v} &\leq \sum_{a_v \geq \frac{\pi}{\alpha}} \alpha a_v e^{-\alpha a_v} \\
 &\leq \sum_{a_v \geq \frac{\pi}{\alpha}} \alpha e^{\frac{C}{2} v} e^{-\alpha e^{\frac{Cv}{2}}} \\
 &\leq \frac{2}{C} \int_0^{\infty} e^{-y} dy = O(1) .
 \end{aligned}$$

Thus $A(\frac{\pi}{\alpha}) > C f(\alpha)$, for every $C > 0$, as $\alpha \rightarrow 0$.

(2.4.9) Remark: In view of theorem (2.1.8) and (2.1.13) asymptotic expansion may be determined for those A for which $\lim_{v \rightarrow \infty} \frac{\log a_v}{v} > 0$ and for which $A(\frac{\pi}{\alpha f^{\delta}(\alpha)}) > A^{\epsilon}(\frac{\pi}{\alpha})$, where δ and ϵ are positive constants.

Thus, for example, for those A for which

$$1 \leq K_1(A) \leq K_2(A) < \infty .$$

CHAPTER III

Some Applications

§3.1 Mahler's Problem

We apply our general theorem (2.1.13) to Mahler's problem in §3.1 and to related problems in §3.2. Recalling that $a_v = r^v$, $v = 0, 1, \dots$ in Mahler's problem we see that theorem (2.4.5) is really sufficient since clearly $A(2u) \leq 2A(u)$ for all $u \geq 1$ and

$$\begin{aligned} f(\alpha) &\leq \sum_{v=0}^{\infty} e^{-\alpha r^v} \leq \int_{-1}^{\infty} e^{-\alpha r^x} dx \\ &= \frac{1}{\log r} \int_{\alpha}^{\infty} \frac{e^{-y}}{y} dy < \frac{1}{\log r} \int_{\alpha r^{-1}}^1 \frac{1}{y} dy + O(1) \\ &= O(\log \frac{1}{\alpha}) . \end{aligned}$$

Thus the asymptotic expansion can be obtained in terms of α where α is defined by

$$(3.1.1) \quad n = \sum_{v=0}^{\infty} \frac{r^v}{e^{\alpha r^v} - 1} .$$

Let us therefore begin by solving (3.1.1), that is expressing α in terms of elementary functions of n .

Throughout this chapter we shall employ the Euler-Maclaurin formula. This states [see [18], p. 13] that:

Let $\phi(x)$ be any function with a continuous derivative in the interval (a,b) . Then, if $[x]$ denotes the greatest integer not exceeding x ,

(3.1.2) (Euler-Maclaurin)

$$\sum_{a < n \leq b} \phi(n) = \int_a^b \phi(x) dx + \int_a^b (x - [x] - \frac{1}{2}) \phi'(x) dx + \\ + (a - [a] - \frac{1}{2}) \phi(a) - (b - [b] - \frac{1}{2}) \phi(b) .$$

If we apply (3.1.2) to (3.1.1) we obtain

$$\sum_{v=0}^{\infty} \frac{r^v}{e^{\alpha r^v} - 1} = \int_0^{\infty} \frac{r^x}{e^{\alpha r^x} - 1} dx + \frac{1}{e^{\alpha} - 1} + \\ + \log r \int_0^{\infty} (x - [x]) \frac{e^{\alpha r^x} - 1 - \alpha r^x}{(e^{\alpha r^x} - 1)^2} e^{\alpha r^x} r^x dx .$$

We let $r^x = y$, $h(y) = y(e^y - 1)$, thus $dy = y \log r dx$ and

$$(3.1.3) \quad n = \int_0^{\infty} r^x (e^{\alpha r^x} - 1) dx + (e^{\alpha} - 1)^{-1} + \frac{1}{\alpha} \int_{\alpha}^{\infty} (x - [x]) h'(y) dy .$$

Now

$$\int_{\alpha}^{\infty} (x - [x]) h'(y) dy = \int_{\alpha}^{\infty} ((\log_r y + \log_r \frac{1}{\alpha})) h'(y) dy$$

where we let $((x)) = x - [x]$.

Since

$$\begin{aligned} & \int_{\alpha}^{\infty} ((\log_r y + \log_r \frac{1}{\alpha})) h'(y) dy = \\ & = \int_0^{\infty} ((\log_r y + \log_r \frac{1}{\alpha})) h'(y) dy - \int_0^{\alpha} ((\log_r y + \log_r \frac{1}{\alpha})) h'(y) dy , \end{aligned}$$

if we let

$$\int_0^{\infty} h'(y) ((\log_r y + \log_r \frac{1}{\alpha})) dy = \psi(\log_r \frac{1}{\alpha}) ,$$

then ψ is a periodic function of period 1 . Now

$$\begin{aligned} \int_0^{\alpha} ((\log_r y + \log_r \frac{1}{\alpha})) h'(y) dy &= \int_0^{\alpha} ((\log_r y + \log_r \frac{1}{\alpha})) (1+O(\alpha)) dy = \\ &= \int_0^{\alpha} ((\log_r y + \log_r \frac{1}{\alpha})) dy + O(\alpha^2) . \end{aligned}$$

Also

$$\int_0^{\alpha} ((\log_r \frac{y}{\alpha})) dy = \alpha \log r \int_{-\infty}^0 ((u)) r^u du ,$$

thus

$$\begin{aligned} (3.1.5) \quad \frac{1}{\alpha} \int_{\alpha}^{\infty} ((\log_r y + \log_r \frac{1}{\alpha})) h'(y) dy &= \frac{1}{\alpha} \psi(\log_r \frac{1}{\alpha}) - \int_{-\infty}^0 ((u)) r^u du \\ &+ O(\alpha) . \end{aligned}$$

Furthermore one can easily show that

$$(3.1.6) \quad \int_0^{\infty} \frac{r^x}{e^{\alpha r^x} - 1} dx = (\alpha \log r)^{-1} \log \frac{1}{\alpha} + \frac{1}{2 \log r} + O(\alpha) .$$

From (3.1.3), (3.1.5) and (3.1.6) we obtain

(3.1.7) Lemma:

$$(3.1.8) \quad n = \frac{1}{\alpha} \left\{ \log_r \frac{1}{\alpha} + \psi_1 \left(\log_r \frac{1}{\alpha} \right) \right\} + A + O(\alpha) \quad ,$$

where

$$A = -\frac{1}{2} - \int_{\infty}^0 ((t)) r^t dt + \frac{1}{2 \log r} \quad ,$$

and

$$\psi_1(x) = 1 + \psi(x) \quad ,$$

$\psi(x)$ being defined by (3.1.4) .

We write

$$(3.1.9) \quad \frac{1}{\alpha} = \frac{m}{\log_r m} h(m) \quad , \quad m = n-A$$

and substitute this in (3.1.8) thus obtaining

$$\frac{1}{h(m)} + O\left(\frac{\alpha}{m}\right) = 1 - \frac{\log_r^{(2)} m}{\log_r m} + \frac{\log_r h(m)}{\log_r m} + \frac{\psi_1 \left(\log_r \frac{1}{\alpha} \right)}{\log_r m} \quad ,$$

(where $\log_r^{(2)} m \equiv \log_r \log_r m$) .

Hence we obtain that

$$\log h(m) = O \left(\frac{\log_r^{(2)} m}{\log_r m} \right) \quad .$$

From this we conclude that (by 3.1.8)

$$(3.1.10) \quad \alpha = \frac{1}{m} \left\{ \log_r \left(\frac{m}{\log_r m} \right) + \psi_1 \left(\log_r \frac{1}{\alpha} \right) \right\} + O \left(\frac{\log_r^{(2)} m}{m \log_r m} \right).$$

(3.1.11) Lemma:

$$\psi(x) = \int_0^\infty ((\log_r y + x)) h'(y) dy$$

is a uniformly continuous function of x on $[0, \infty)$.

Proof: Let
$$\psi(x) = \int_0^\infty ((\log_r y + x)) h'(y) dy$$
$$= \log r \int_{-\infty}^\infty ((s+x)) h'(r^s) r^s ds.$$

Now, upon integrating by parts,

$$\begin{aligned} \int_{-\infty}^\infty ((s+x)) h'(r^s) r^s ds &= \log r \int_{-\infty}^\infty (((s+x))) h''(r^s) r^{2s} ds + \\ &+ \log r \int_{-\infty}^\infty (((s+x))) h'(r^s) r^s ds \end{aligned}$$

where
$$(((s+x))) = \sum_{n=1}^\infty \frac{2 \cos n \pi(s+x)}{(2n\pi)^2}.$$

Since this series converges uniformly;

$$\begin{aligned} (3.1.12) \quad & - \log r \int_{-\infty}^\infty (((s+x))) h'(r^s) r^s ds = \\ &= - \log r \int_{-\infty}^\infty \left(\sum_{n=1}^\infty \frac{2 \cos 2n \pi(s+x)}{(2n \pi)^2} \right) h'(r^s) r^s ds \\ &= - \log r \sum_{n=1}^\infty \int_{-\infty}^\infty \frac{2 \cos 2n \pi(s+x)}{(2n \pi)^2} h'(r^s) r^s ds. \end{aligned}$$

Now $h'(r^s) r^s$ goes to zero exponentially as $s \rightarrow \infty$ or $-\infty$,
thus

$$\int_{-\infty}^{\infty} h'(r^s) r^s < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{\cos 2n \pi x}{(2n \pi)^2} \int_{-\infty}^{\infty} \cos (2n \pi s) h'(r^s) r^s ds ,$$

$$\sum_{n=1}^{\infty} \frac{\sin 2n \pi x}{(2n \pi)^2} \int_{-\infty}^{\infty} \sin (2n \pi s) h'(r^s) r^s ds$$

are sums of absolutely and uniformly convergent Fourier series. Since $\cos 2n \pi x$ and $\sin 2n \pi x$ are uniformly convergent on $[0, \infty)$, so must these sums be. This proves that the left-hand side of (3.1.12) is uniformly convergent on $[0, \infty)$ hence we have proved the lemma.

Previously we obtained that

$$\frac{1}{\alpha} = m h(m) / \log_r m$$

where

$$h(m) = 1 - \frac{\log_r^{(2)} m}{\log_r m} + \frac{\psi_1(\log_r \frac{1}{m})}{\log_r m} + O\left\{ \left(\frac{\log_r^{(2)} m}{\log_r m} \right)^2 \right\} .$$

Hence $\log_r \frac{1}{\alpha} = \log_r m - \log_r^{(2)} m + \log_r h(m)$, but since

$$\log_r h(m) = \frac{\log_r^{(2)} m}{\log_r m} - \frac{\psi_1(\log_r(\frac{m}{\log_r m}) + \log_r h(m))}{\log_r m} + O\left(\frac{\log_r^{(2)} m}{\log_r m}\right)^2$$

we have by lemma (3.1.11)

$$\log_r \frac{1}{\alpha} = \log_r\left(\frac{m}{\log_r m}\right) - \frac{\log_r^{(2)} m + \psi_1(\log_r(\frac{m}{\log_r m}))}{\log_r m} + O\left\{\left(\frac{\log_r^{(2)} m}{\log_r m}\right)^2\right\}.$$

This with (3.1.9) yield, after another application of lemma (3.1.11),

$$(3.1.13) \quad \frac{1}{\alpha} = \frac{m}{\log_r m} \left\{ 1 + \frac{P_1(m)}{\log_r m} + O\left(\frac{\log_r^{(2)} m}{\log_r m}\right)^2 \right\},$$

where $P_1(m) = \psi_1(\log_r(\frac{m}{\log_r m})) - \log_r^{(2)} m$.

We now prove

(3.1.14) Theorem:

$$\frac{1}{\alpha} = \frac{m}{\log_r m} \left\{ 1 + \frac{P_1(m)}{\log_r m} + \frac{P_2(m)}{(\log_r m)^2} + \dots + \frac{P_j(m)}{(\log_r m)^j} + O\left(\frac{\log_r^{(2)} m}{\log_r m}\right)^{j+1} \right\}$$

where $P_j(m) = O\{(\log_r^{(2)} m)^j\}$ and P_j is defined recursively in terms of $P_1, P_2, \dots, P_{j-1}$.

Proof: The proof is by induction. We prove it now for $j = 2$ using the fact that it is true for $j = 1$ (see (3.1.13)). The step from j to $j+1$ is completely analogous hence for reasons of space we omit it. From (3.1.13)

$$\frac{1}{\alpha} = \frac{m}{\log_r m} \left[1 + \frac{P_1(m)}{\log_r m} + f(m) \right] , \quad \text{where}$$

$$f(m) = O\left(\frac{\log_r^{(2)} m}{\log_r m}\right)^2 .$$

Also, from (3.1.8) and (3.1.9),

$$1 = \frac{1}{h(m)} \log_r m \left(\log \frac{1}{\alpha} + \psi\left(\log_r \frac{1}{\alpha}\right) \right) + O\left(\frac{\alpha}{m}\right) .$$

Now

$$\log_r \frac{1}{\alpha} = \log_r \left(\frac{m}{\log_r m} \right) + \frac{P_1(m)}{\log_r m} + f(m) + O\left(\frac{\log_r^{(2)} m}{\log_r m}\right)^3 ,$$

thus by lemma (3.1.11)

$$\psi_1\left(\log_r \frac{1}{\alpha}\right) = \psi_1\left(\log_r \left(\frac{m}{\log_r m}\right) + \frac{P_1(m)}{\log_r m}\right) + O\left(\frac{\log_r^{(2)} m}{\log_r m}\right)^3 ,$$

and

$$\begin{aligned} h(m) = 1 - \frac{\log_r^{(2)} m}{\log_r m} + \frac{P_1(m)}{(\log_r m)^2} + \psi_1\left(\log\left(\frac{m}{\log_r m}\right) + \frac{P_1(m)}{\log_r m}\right) + \\ + O\left(\frac{\log_r^{(2)} m}{\log_r m}\right)^3 . \end{aligned}$$

However

$$h(m) = 1 + \frac{P_1(m)}{\log_r m} + f(m) ,$$

thus

$$f(m) = \frac{\psi_1(\log_r(\frac{m}{\log_r m}) + \frac{P_1(m)}{\log_r m}) - \log_r^{(2)} m - P_1(m)}{\log_r m} + \frac{P_1(m)}{(\log_r m)^2} +$$

$$+ O(\frac{\log_r^{(2)} m}{\log_r m})^3 = \frac{P_2(m)}{(\log_r m)^2} + O(\frac{\log_r^{(2)} m}{\log_r m})^3 .$$

As a check, we note that

$$\frac{\psi_1(\log_r(\frac{m}{\log_r m}) + \frac{P_1(m)}{\log_r m})}{\log_r m} = \frac{\psi_1(\frac{\log_r m}{\log_r m})}{\log_r m} + O(\frac{P_1(m)}{(\log_r m)^2})$$

and that $\psi_1(\log_r(\frac{m}{\log_r m})) - \log_r^{(2)} m - P_1(m) = 0$, thus $f(m) =$

$O(\log_r^{(2)} m / \log_r m)^2$. This completes the proof of the theorem.

Since

$$\frac{1}{m} = \frac{1}{n} + O(\frac{1}{n})^2$$

$$\log_r m = \log_r n + O(\frac{1}{n})$$

theorem (3.1.14) can be rewritten as

$$(3.1.15) \quad \alpha n = \log_r n + P_1(n) + \frac{P_2(n)}{\log_r n} + \dots + \frac{P_j(n)}{(\log_r n)^{j-1}} + O\left\{\frac{(\log_r n)^{j+1}}{(\log_r n)^j}\right\}$$

Since

$$\log F(\rho) = - \sum_{v=0}^{\infty} \log (1 - e^{-\alpha r^v})$$

we could apply the Euler-Maclaurin summation formula (see 3.1.2) to obtain

$$\begin{aligned} \log F(\rho) &= \frac{1}{2} \frac{\log^2 \frac{1}{\alpha}}{\log r} + \frac{1}{\log r} \int_0^\infty \frac{\log y - \log(1 - e^{-y})}{e^y - 1} dy + \\ &- \frac{1}{2} \log(1 - e^{-\alpha}) + \frac{\log r}{12} - \log r \int_0^\infty (((\log_r y + \log_r \frac{1}{\alpha}))) h'(y) dy \\ &+ O(\alpha \log \frac{1}{\alpha}) . \end{aligned}$$

We need not do this since W.B. Pennington [11] shows, by a method which we shall describe in §3.2, that

(3.1.16)

$$\begin{aligned} \log F(\rho) &= \frac{1}{2 \log r} \log^2 \frac{1}{\alpha} + \frac{1}{2} \log \frac{1}{\alpha} + \left\{ \frac{\log r}{12} - \frac{1}{\log r} \int_0^\infty \frac{\log y - \log(1 - e^{-y})}{e^y - 1} \right\} \\ &+ \frac{1}{\log r} \sum_{\substack{v=-\infty \\ v \neq 0}}^\infty \left(\frac{2\pi i v}{\log r} \right) \zeta \left(1 + \frac{2\pi i v}{\log r} \right) e^{2\pi i v \log_r \frac{1}{\alpha}} + O(\alpha^{\frac{1}{2}}) . \end{aligned}$$

He shows that the Fourier series is uniformly convergent on $[0, \infty)$.

We now show that:

(3.1.17) Lemma: Let A_2 be defined as in theorem (2.1.1). Then

$$A_2 = \frac{\alpha^{-2}}{\log r} \left\{ \log_r \frac{1}{\alpha} + \psi_3 \left(\log_r \frac{1}{\alpha} \right) + O(\alpha) \right\} ,$$

where

$$\psi_3(x) = 1 + \int_0^\infty ((\log_r y + x)) \left(\frac{y^2 e^y}{(e^y - 1)^2} \right)' dy .$$

Proof:

$$A_2 = \alpha^{-2} \sum_{v=0}^{\infty} (\alpha r^v)^2 \frac{e^{\alpha r^v}}{(e^{\alpha r^v} - 1)^2}$$

Thus

$$\begin{aligned} A_2 &= \alpha^{-2} \int_0^{\infty} \frac{(\alpha r^x)^2 e^{\alpha r^x}}{(e^{\alpha r^x} - 1)^2} dx + \alpha^{-2} \frac{\alpha^2}{2} \frac{e^{\alpha}}{(e^{\alpha} - 1)^2} + \\ &+ \alpha^{-2} \int_0^{\infty} ((x)) \left[\frac{(\alpha r^x)^2 e^{\alpha r^x}}{(e^{\alpha r^x} - 1)^2} \right]' dx, \end{aligned}$$

by the Euler-Maclaurin summation formula (see (3.1.2)). Thus

$$\begin{aligned} A_2 &= \frac{\alpha^{-2}}{\log r} \int_{\alpha}^{\infty} \frac{e^y y}{(e^y - 1)^2} dy + \frac{e^{\alpha}}{2(e^{\alpha} - 1)^2} + \\ &+ \alpha^{-2} \int_{\alpha}^{\infty} ((\log_r \frac{y}{\alpha})) \left[\frac{y^2 e^y}{(e^y - 1)^2} \right]' dy \\ &= \frac{\alpha^{-2}}{\log r} \left[\int_{\alpha}^{\infty} \frac{1}{e^y - 1} + \frac{\alpha}{e^{\alpha} - 1} \right] + \frac{e^{\alpha}}{2(e^{\alpha} - 1)^2} + \\ &+ \alpha^{-2} \int_{\alpha}^{\infty} ((\log_r \frac{y}{\alpha})) \left[\frac{y^2 e^y}{(e^y - 1)^2} \right]' dy. \end{aligned}$$

If we treat this last integral as in lemma (3.1.7) we obtain our theorem.

We may now use lemma (3.1.17), (3.1.16), and (3.1.15) to obtain the asymptotic formula of Mahler's problem. We shall calculate the principle term only because while additional terms may easily be calculated

in principle the calculations involved are prohibitively long and cumbersome. Moreover there does not seem to be any simple formula for the higher terms. We shall show however that our result is the same as Pennington's [11] and de Bruijn's [16].

We readily obtain from (3.1.15) that

$$(3.1.18) \quad n^\alpha = \log_r n = P_1(n) + O\left(\frac{\log_r^{(2)} n}{\log_r n}\right)^2,$$

thus

$$(3.1.19) \quad \log \frac{1}{\alpha} = \log \left(\frac{n}{\log_r n}\right) - \frac{P_1(n)}{\log_r n} + O\left(\frac{\log_r^{(2)} n}{\log_r n}\right)^2$$

and

$$(3.1.20) \quad \left(\log \frac{1}{\alpha}\right)^2 = \log^2 \left(\frac{n}{\log_r n}\right) - 2 \log_r P_1(n) + O\left\{\frac{(\log_r^{(2)} n)^2}{\log_r n}\right\}.$$

We use the formula for $P(n)$ in theorem (2.1.13). From this it follows that

$$\log P(n) = n^\alpha + \log F(e^{-\alpha}) - \frac{1}{2} A_2 - \frac{1}{2} \log 2\pi + O(\log n)^{-\frac{4}{3}}.$$

Thus from (3.1.18), (3.1.20), (3.1.16) and lemma (3.1.17)

$$\begin{aligned} \log P(n) &= \frac{1}{2 \log r} \left\{ \log \left(\frac{n}{\log_r n} \right) \right\}^2 + \left\{ \frac{1}{\log r} - \frac{1}{2} \right\} \log n = \frac{1}{2} \log \log_r n \\ &+ \frac{1}{2} \log \log n + \left\{ \frac{\log 2\pi}{2} + \frac{\log r}{12} - \frac{1}{2} \log \log r - \frac{1}{\log r} \right. \\ &\left. \int_0^\infty \frac{\log y - \log(1 - e^{-y})}{e^y - 1} dy \right\} \\ &+ \phi \left(\log_r \left(\frac{n}{\log_r n} \right) \right) + O(\log n)^{-\frac{4}{3}}, \end{aligned}$$

where

$$\phi(x) = \frac{1}{\log r} \sum_{\substack{v=-\infty \\ v \neq 0}}^{\infty} \Gamma\left(\frac{2-i}{\log r}\right) \zeta\left(1 + \frac{2\pi i v}{\log r}\right) e^{2\pi i v x}.$$

Thus

$$\begin{aligned} \log P(r n) &= \frac{1}{2 \log r} (\log n - \log_n^{(2)} + \log_r^{(2)})^2 + \\ &+ \left(\frac{1}{\log r} + \frac{1}{2} \right) \log n - \log \log n + \log_r^{(2)} + \\ &+ \phi \left(\frac{\log n - \log_n^{(2)} + \log_r^{(2)}}{\log r} \right) \\ &+ O \left\{ \frac{(\log_n^{(2)})^2}{\log n} \right\}. \end{aligned}$$

We see that this is Pennington's result with $o(1)$ replaced by

$O \left\{ \frac{(\log_n^{(2)})^2}{\log n} \right\}$; which he shows is equivalent to de Bruijn's result.

§3.2 Partitions into Numbers Related to the Fibonacci numbers.

In this section we determine the asymptotic expansion of $P_A(n)$ when the n -th element a_n of A is given by

$$a_n = \frac{r^n - r^{-n}}{d} \quad .$$

Those sequences for which

$$a_n = \frac{r^{+n} + r^{-n}}{d}$$

may be treated similarly although there are small differences. We consider the Fibonacci numbers $1, 2, 3, 5, \dots$ as

$$F_n = \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - (-1)^n \left(\frac{1 + \sqrt{5}}{2} \right)^{-n} \right] / \sqrt{5} \quad , \quad n=2, 3, 4, \dots \quad .$$

Although the Fibonacci numbers satisfy the conditions of theorem (2.4.5), we shall here only treat the case where $a_n = F_{2n}$. The case $a_n = F_n$ requires additional investigation; however it appears that no new techniques are required. Thus we must solve

$$n = \sum_{n=1}^{\infty} \frac{r^n - r^{-n}}{\sqrt{5}} e^{\alpha \left(\frac{r^n - r^{-n}}{\sqrt{5}} \right) - 1} \quad .$$

We apply the Euler-Maclaurin formula to

$$n = \sum_{n=1}^{\infty} \frac{2}{\sqrt{5}} \sinh(n \log r) \left(\frac{\frac{2\alpha}{\sqrt{5}} \sinh(n \log r)}{e^{\frac{2\alpha}{\sqrt{5}} \sinh(n \log r)} - 1} \right)^{-1}$$

to obtain

$$(3.2.1) \quad n = \frac{2}{\sqrt{5}} \int_1^{\infty} \frac{\sinh(x \log r)}{\frac{\frac{2\alpha}{\sqrt{5}} \sinh(x \log r)}{e^{\frac{2\alpha}{\sqrt{5}} \sinh(x \log r)} - 1}} dx + \frac{2}{\sqrt{5}} \frac{\sinh(\log r)}{\frac{\frac{2\alpha}{\sqrt{5}} \sinh(\log r)}{e^{\frac{2\alpha}{\sqrt{5}} \sinh(\log r)} - 1}} \\ + \frac{2}{\sqrt{5}} \int_1^{\infty} ((x)) \frac{d}{dx} \left(\frac{\sinh(x \log r)}{\frac{\frac{2\alpha}{\sqrt{5}} \sinh(x \log r)}{e^{\frac{2\alpha}{\sqrt{5}} \sinh(x \log r)} - 1}} \right) dx .$$

We let $C = 2\alpha/\sqrt{5}$, then

$$(3.2.2) \quad \int_1^{\infty} \sinh(x \log r) (e^{C \sinh(x \log r)} - 1)^{-1} dx = \\ = \frac{1}{\log r} \int_{1 \log r}^{\infty} \frac{\sinh y}{e^{C \sinh y} - 1} dy .$$

We let $C \sinh y = z$, then

$$(3.2.3) \quad \int_{1 \log r}^{\infty} \frac{\sinh y}{e^{C \sinh y} - 1} = \frac{1}{C} \int_{C \sinh(\log r)}^{\infty} \frac{z}{\sqrt{z^2 + C^2}} \frac{1}{e^z - 1} dz .$$

We break the integral on the right-hand side up into two integrals, i.e.

$$\int_{\frac{3}{C}}^{\infty} \frac{1}{\sinh(\log r)} \frac{z}{\sqrt{z^2 + C^2}} \frac{1}{e^z - 1} dz = \int_{\frac{3}{C}}^{\frac{3}{C}} \frac{1}{\sinh(\log r)} + \int_{\frac{3}{C}}^{\infty} =$$

$$= I_1 + I_2 .$$

On I_1

$$\frac{z}{e^z - 1} = 1 - \frac{1}{2} z + \frac{1}{12} z^2 + \frac{1}{720} z^4 + o(C^{\frac{9}{4}}) .$$

Thus (3.2.4)

$$\frac{1}{C} I_1 = \frac{1}{C} \int_{\frac{3}{C}}^{\frac{3}{C}} \frac{1 - \frac{1}{2} z + \frac{1}{12} z^2 + \frac{1}{720} z^4}{\sqrt{z^2 + C^2}} dz + o(C^{\frac{1}{4}}) .$$

For I_2 since

$$\frac{z}{\sqrt{z^2 + C^2}} = 1 - C^2 \left(z + \sqrt{z^2 + C^2} \right)^{-1} (z^2 + C^2)^{-\frac{1}{2}}$$

$$(3.2.5) \quad \frac{1}{C} I_2 = \frac{1}{C} \int_{\frac{3}{C}}^{\infty} \frac{1}{e^z - 1} dz + o \left(C^{\frac{1}{4}} \int_{\frac{3}{C}}^{\infty} \frac{1}{e^z - 1} dz \right)$$

From (3.2.5) we obtain

$$(3.2.6) \quad \frac{1}{C} I_2 = -\frac{1}{C} \log(1 - e^{-\frac{3}{C}}) + o(C^{\frac{1}{4}} \log C) =$$

$$= \frac{3}{8} \log \frac{1}{C} + \frac{1}{2} C^{-\frac{5}{8}} - \frac{1}{24} C^{-\frac{1}{4}} + o(C^{\frac{1}{8}}) .$$

From (3.2.4)

(3.2.7)

$$\begin{aligned} \frac{1}{C} I_1 &= \frac{1}{C} \left\{ \frac{3}{8} \log C - \log C + \log 2 - \log[\sinh(\log r) + \cosh(\log r)] \right\} \\ &\quad - \frac{1}{2} C^{-\frac{5}{8}} + \frac{1}{24} C^{-\frac{1}{4}} + \frac{\cosh(\log r)}{2} + O(C^{\frac{1}{4}}) . \end{aligned}$$

From (3.2.6) and (3.2.7) we obtain

$$\begin{aligned} \frac{1}{\log r} \int_1^\infty \frac{\sinh y}{e^{\frac{1}{C} \sinh y - 1}} dy &= \frac{1}{C \log r} \left[\log \frac{1}{C} + \right. \\ &\quad \left. + \log \frac{2}{\cosh(\log r) + \sinh(\log r)} \right] \\ &\quad + \frac{\cosh(\log r)}{2} + O(C^{\frac{1}{8}}) . \end{aligned}$$

From (3.2.2) we thus obtain that

$$\begin{aligned} (3.2.8) \quad \int_1^\infty \sinh(x \log r) (e^{\frac{1}{C} \sinh(x \log r) - 1})^{-1} dx &= , \\ &= \frac{1}{C \log r} \left[\log \frac{1}{C} + \log \frac{2}{\sinh(\log r) + \cosh(\log r)} \right] + \\ &\quad + \frac{\cosh(\log r)}{1} + O(C^{\frac{1}{8}}) . \end{aligned}$$

Moreover

$$(3.2.9) \quad \frac{\sinh(\log r)}{e^{\frac{C}{2} \sinh(\log r)} - 1} = \frac{1}{C} - \frac{\sinh(\log r)}{2} + O(C) \quad .$$

Thus substituting (3.2.9) and (3.2.8) into (3.2.1) yields

(3.2.10)

$$\begin{aligned} n = & \frac{2}{C\sqrt{5} \log r} \left[\log \frac{1}{C} + 1 + \log 2(\sinh(\log r) + \cosh(\log r))^{-1} \right] + \\ & + \frac{1}{\sqrt{5}} (\cosh(\log r) - \sinh(\log r)) + \\ & + \frac{2}{\sqrt{5}} \int_1^\infty ((x)) \frac{d}{dx} \left(\frac{\sinh(x \log r)}{e^{\frac{C}{2} \sinh(x \log r)} - 1} \right) dx + O(C^{\frac{1}{8}}) \quad . \end{aligned}$$

Let

$$\begin{aligned} I_3 &= \int_1^\infty ((x)) \frac{d}{dx} \left(\frac{\sinh(x \log r)}{e^{\frac{C}{2} \sinh(x \log r)} - 1} \right) dx \\ &= \frac{1}{C \log r} \int_{\sinh(\log r)}^\infty \left(\frac{(\sinh^{-1} z)}{\log r} \right) \frac{d}{dz} \left(\frac{C z}{e^{Cz} - 1} \right) dz \quad . \end{aligned}$$

We use the formula (that Pennington [11] also uses) stated below to treat I_1 . For $\sigma = \operatorname{Re} C > 0$, $u > 0$

$$-\frac{\partial}{\partial z} \frac{Cz}{e^{Cz} - 1} = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \Gamma(t) \{(t-1)\zeta(t)\} C^{1-t} z^{-t} dt \quad .$$

This formula follows from Mellin's integral formula

$$e^{-\omega} = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \Gamma(z) \omega^{-z} dz \quad (\beta > 0, \omega > 0) .$$

Let $1 < \beta = \operatorname{Re} t \leq 2$. Then

$$\begin{aligned} I_1 &= \frac{1}{C} \frac{1}{2\pi i} \log r \int_{\sinh(\log r)}^{\infty} \left(\left| \frac{\sinh^{-1} z}{\log r} \right| \right) \int_{\beta-i\infty}^{\beta+i\infty} \Gamma(t) \{(t-1)\zeta(t)\} C^{-t} z^{-t} dt \\ &= \frac{1}{C} \frac{1}{2\pi i} \log r \int_{\beta-i\infty}^{\beta+i\infty} \Gamma(t) \{(t-1)\zeta(t)\} C^{1-t} dt \int_{\sinh(\log r)}^{\infty} \left(\left| \frac{\sinh^{-1} z}{\log r} \right| \right) z^{-t} dt \end{aligned}$$

The inversion is justified here because $\beta > 1$, $((\sinh^{-1} z / \log r))$ is bounded, and

$$\begin{aligned} \zeta(x+iy) &= O\{|y|^{\frac{1}{2}}\} & (\frac{1}{2} \leq x \leq 2, |y| \geq 1) , \\ (3.2.11) \quad \Gamma(x+iy) &= O\{e^{-\frac{\pi}{2}|y|} |y|^{\frac{3}{2}}\} & (-1 \leq x \leq 2) . \end{aligned}$$

(See Titchmarsh [19], p. 151 and [18], p. 81) .

Let us consider

$$\int_{\sinh(\log r)}^{\infty} \left(\left| \frac{\sinh^{-1} z}{\log r} \right| \right) z^{-t} dz =$$

$$= \log r \int_1^{\infty} ((u)) \{ \sinh(u \log r) \}^{-t} \cosh(u \log r) \, du .$$

Now $((u))$ has Fourier coefficients $C_v = \frac{1}{2\pi i v}$, $v \neq 0$, $C_0 = 0$ and the function $\sinh^{-t}(u \log r) \cosh(u \log r)$ is of bounded variation in any finite interval $0 < \xi \leq u \log r \leq \eta$. W.H. Young [20] has proven that (see also A. Zygmund [21]): (Young). If one of the two functions $f(x)$ and $g(x)$ has bounded variation, while the other is summable, the integral of their product between any finite limits may be evaluated by term-by-term integration of the series obtained by multiplying the Fourier series of either term by term by the other function, provided only that, if the length of the interval of integration exceeds 2π , the function whose Fourier series is employed is periodic. This is the statement of the theorem which we shall call Young's theorem.

Let $\sum'$ denote summation over nonzero values of the index.

Then by Young's theorem;

$$\begin{aligned} & \log r \int_{\xi}^{\eta} ((u)) \{ \sinh(u \log r) \}^{-t} \cosh(u \log r) \, du = \\ &= \log r \sum_{v=-\infty}^{\infty} \frac{1}{2\pi i v} \int_{\xi}^{\eta} e^{2\pi i v u} \{ \sinh(u \log r) \}^{-t} \cosh(u \log r) \, du = \\ &= 2^{t-1} \log r \sum_{v=-\infty}^{\infty} \frac{1}{2\pi i v} \int_{\xi}^{\eta} e^{[2\pi i v - (t-1)\log r]u} \{ 1 - e^{-2u \log r} \}^{-t} \\ & \qquad \qquad \qquad (1 + e^{-2u \log r}) \, du . \end{aligned}$$

Let us now consider

$$I_4 = \int_{\xi}^{\eta} e^{[2\pi i v - (t-1)\log r]u} \{1 - e^{-2u \log r}\}^{-t} (1 + e^{-2u \log r}) du .$$

Since $0 < e^{-2u \log r} \leq e^{-1 \log r} < 1$,

$$\{1 - e^{-2u \log r}\}^{-t} (1 + e^{-2u \log r}) = \sum_{\ell=0}^{\infty} b_{\ell} e^{-2u \ell \log r} ,$$

where

$$b_{\ell} = a_{\ell} + a_{\ell-1}$$

and

$$a_{\ell} = (-1)^{\ell} \frac{t(t-1)(t-2)\cdots(t-\ell)}{\ell!} .$$

Since this series converges absolutely and uniformly ($u \geq$) we can integrate term by term obtaining

$$I_4 = \frac{e^{[2\pi i v - (t-1)\log r]\eta} - e^{[2\pi i v - (t-1)\log r]\xi}}{2\pi i v - (t-1) \log r} + \sum_{\ell=1}^{\infty} b_{\ell} \frac{e^{[2\pi i v - (t+2\ell-1)\log r]\eta} - e^{[2\pi i v - (t+2\ell-1)\log r]\xi}}{2\pi i v - (t-1) \log r} .$$

Since $\sum_{v=-\infty}^{\infty} \frac{1}{v^2} < \infty$ and $\operatorname{Re} t = \beta > 1$ we can let $\eta \rightarrow \infty$ and setting

$\xi = 1$ we get

$$\begin{aligned} v(t) &= \log r \int_1^{\infty} ((u)) \{\sinh(u \log r)\}^{-t} \cosh(u \log r) du = \\ &= \sum_{v=-\infty}^{\infty} \frac{\log v}{2\pi i v} 2^{t-1} \frac{e^{[2\pi i v - (t-1) \log r]}}{t-1-2\pi i v} + \end{aligned}$$

$$+ \sum_{v=-\infty}^{\infty} \frac{\log r \cdot 2^{t-1}}{2\pi i v} \sum_{\ell=1}^{\infty} b_{\ell} \frac{e^{[2\pi i v - (t+2\ell-1) \log r]}}{t + 2\ell - 1 - \frac{2\pi i v}{\log r}} .$$

Since we have absolute convergence we can interchange the order of summation in the double sum. Then we obtain

$$v(t) = \sum_{\ell=0}^{\infty} v_{\ell}(t)$$

where

$$v_{\ell}(t) = b_{\ell} \sum_{v=-\infty}^{\infty} \frac{\log r \cdot 2^{t-1}}{2\pi i v} \frac{e^{[2\pi i v - (t+2\ell-1) \log r]}}{t + 2\ell - 1 - \frac{2\pi i v}{\log r}} .$$

Since $\sum_{v=+\infty}^{\infty} \frac{1}{v^2} < \infty$ we easily see that $v_{\ell}(t)$ is analytic in the entire

plane with simple poles at $1 - 2\ell + (2\pi i v)/\log r$, $v = \pm 1, \pm 2, \dots$.

Since $|v_{\ell}(t)| = O\{|b_{\ell}| e^{-\ell \log r}\}$ and the 0-constant is independent of ℓ we obtain that $v(t) = \sum_{\ell=0}^{\infty} v_{\ell}(t)$ is analytic in the entire plane

with simple poles at the points

$$t_{\ell}^v = 1 - 2\ell + \frac{2\pi i v}{\log r} , \quad \begin{array}{l} \ell = 0, 1, 2, \dots \\ v = \pm 1, \pm 2, \dots \end{array}$$

and residue $2^{t_{\ell}^v-1} \log r \cdot b_{\ell} / 2\pi i v$ at t_{ℓ}^v . From (3.2.12)

$$I_3 = \frac{1}{C \cdot 2\pi i \log r} \int_{\beta-i\infty}^{\beta+i\infty} \Gamma(t) \{(t-1)\zeta(t)\} C^{1-t} v(t) dt .$$

We can obtain an estimation of this integral by moving the path of integration from the line $t = \beta$ to $t = -\frac{1}{2}$ say.

On the line $t = \beta$

$$\begin{aligned} v(t) &= O\left\{ \int_1^{\infty} [\sinh(u \log r)]^{-t} \cosh(u \log r) du \right\} = \\ &= O\left\{ \int_{\sinh(\log r)}^{\infty} u^{-\beta} du \right\} \\ &= O(1) , \end{aligned}$$

since $\beta > 1$. Since

$$\begin{aligned} v(t) &= 2^{t-1} \log r \sum_{v=-\infty}^{\infty} \frac{1}{2\pi i v} \int_1^{\infty} e^{[2\pi i v - (t-1) \log r]u} [1 - e^{-2u \log r}]^{-t} \\ &\quad (1 + e^{-2u \log r}) du \\ &= - \sum_{v=-\infty}^{\infty} \frac{2^{t-1} \log r}{2\pi i v} \frac{[1 - e^{-1 \log r}]^{-t} (1 + e^{-1 \log r})}{2\pi i v - (t-1) \log r} \times \\ &\quad \times e^{[2\pi i v - (t-1) \log r]} + \\ &+ -(t+1) \sum_{v=-\infty}^{\infty} \frac{2^{t-1} \log r}{2\pi i v [2\pi i v - (t-1) \log r]} \int_1^{\infty} e^{[2\pi i v - (t-1) \log r] u} \times \\ &\quad \times e^{-2u \log r} (1 - e^{-2u \log r})^{-t} \times (1 + e^{-2u \log r}) du + \\ &+ \sum_{v=-\infty}^{\infty} \frac{2^{t-1} \log r}{2\pi i v [2\pi i v - (t-1) \log r]} \int_1^{\infty} e^{[2\pi i v - (t-1) \log r] u} \\ &\quad e^{-2u \log r} (1 - e^{-2u \log r})^{-t} dt = \end{aligned}$$

$$= 0 \{ |t| \sum_{v=-\infty}^{\infty} \left| \frac{1}{2\pi i v [2\pi i v - (t-1) \log r]} \right| \} ,$$

for $\operatorname{Re} t > -1$.

Now on the line $\operatorname{Re} t = -\frac{1}{2}$

$$\frac{|t - (1 + \frac{2\pi i v}{\log r})|}{2\pi |v| / \log r} \geq \frac{\frac{3}{2}}{|I_m t|} , \quad v = \pm 1, \pm 2, \dots$$

Thus

$$\sum_{v=-\infty}^{\infty} (2\pi i v [2\pi i v - (t-1) \log r])^{-1} = 0 \{ |I_m t| \sum_{v=-\infty}^{\infty} \frac{1}{v^2} \} = 0 \{ |I_m t| \} .$$

We may therefore conclude that $v(t) = 0\{|t|^2\}$ on $\operatorname{Re} t = -\frac{1}{2}$

(indeed for $\operatorname{Re} t > -1$) . Furthermore on the lines $I_m t = \pm I(2N+1)\pi/\rho$

($N=1,2,\dots$)

$$\frac{|t - (1 + \frac{2\pi i v}{\log r})|}{2\pi |v| / \log r} \geq \frac{\pi}{\log r |I_m t|} \quad (v = \pm 1, \pm 2, \dots)$$

and as before $v(t) = 0\{|t|^2\}$. Thus applying Cauchy's theorem to the rectangle formed by the lines

$$\operatorname{Re} t = \frac{1}{2} , \operatorname{Re} t = \beta , I_m t = \pm(2N+1)\pi/\log r ,$$

and using (3.2.11) and that

$$(3.2.13) \quad \zeta(t) = O\{|I_m t|\} \quad , \quad \operatorname{Re} t = -\frac{1}{2}$$

(see Titchmarsh [18], p. 81) we conclude that

$$\begin{aligned} I_3 &= \frac{1}{C^{2\pi i \log r}} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(t) \{(t-1)\zeta(t)\} C^{1-t} v(t) dt + \\ &+ \frac{1}{C} \quad (\text{sum of the residues of } \Gamma(z)(z-1)\zeta(z) C^{1-z} v(z) \\ &\quad \text{in the strip } -\frac{1}{2} < \operatorname{Re}(t) < \beta) . \end{aligned}$$

As $c \rightarrow \infty$, the first integral is

$$O\left\{C^{\frac{1}{2}} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(t) |\zeta(t)| |I_m t|^2 |dt|\right\} = O(C^{\frac{1}{2}}) .$$

(Since $v(t) = O\{(I_m t)^2\}$ and by (3.2.11) and (3.2.13.)

The second term is

$$\begin{aligned} (3.2.14) \quad & \sum_{v=-\infty}^{v=\infty} \frac{2\pi i v}{2 \log r} \Gamma\left(1 + \frac{2\pi i v}{\log r}\right) \frac{2\pi i v}{\log r} \zeta\left(1 + \frac{2\pi i v}{\log r}\right) C^{-\frac{2\pi i v}{\log r}} \\ &= \sum_{v=-\infty}^{\infty} \frac{1}{\log r} \Gamma\left(1 + \frac{2\pi i v}{\log r}\right) \zeta\left(1 + \frac{2\pi i v}{\log r}\right) e^{-\frac{2\pi i v}{\log r} \log \frac{C}{2}} \\ &= \psi_0 \left(\log_r \frac{1}{C} + \log_r \sqrt{5}\right) \\ &= \psi \left(\log_r \frac{1}{C}\right) \end{aligned}$$

where ψ has Fourier coefficients

$$c_v = \frac{1}{\log r} \Gamma\left(1 + \frac{2\pi i v}{\log r}\right) \zeta\left(1 + \frac{2\pi i v}{\log r}\right) 5^{-\frac{\pi i v}{\log r}}.$$

We now conclude from (3.2.10) and (3.2.14) that

$$(3.2.15) \quad n = \frac{1}{\alpha} \left[\log_r \frac{1}{\alpha} + \psi_1 \left(\log_r \frac{1}{\alpha} \right) \right] + \\ + \frac{1}{\sqrt{5}} (\cosh (\log r) - \sinh (\log r)) + O(\alpha^{\frac{1}{8}})$$

where

$$\psi_1 = \frac{1}{\log r} + \log_r [2(\sinh (\log r) + \cosh (\log r))^{-1}] + \frac{\sqrt{5}}{2} \psi$$

(ψ is defined by (3.2.14)). This equation is of the same form as (3.1.7) hence can be solved in the same way. Thus we conclude that (from (3.1.12)), (if we show that ψ_1 is uniformly continuous);

$$\alpha = \frac{\log_r n}{n} + \frac{P_1(n)}{n} + \frac{P_2(n)}{n \log_r m} + O\left\{ \frac{(\log_r^{(2)} n)^3}{(\log_r n)^2} \right\}$$

where P_1 and P_2 are defined as in (3.1.12).

We must now estimate

$$\log F(\rho) = - \sum_{v=0}^{\infty} \log (1 - e^{-\alpha a_v})$$

This is done in essentially the same way that (3.2.15) was derived. Hence we merely indicate the method and the differences which arise.

We begin with the Euler-Maclaurin summation formula (see (3.1.2)) to obtain

(3.2.16)

$$\begin{aligned} -\log F(\rho) &= \int_1^{\infty} \log(1 - e^{-C \sinh(x \log r)}) dx + \log(1 - e^{-C \sinh(\log r)}) \\ &\quad + \log r \int_1^{\infty} ((x)) \frac{e^{-C \sinh(x \log r)}}{1 - e^{-C \sinh(x \log r)}} C \cosh(x \log r) dx . \end{aligned}$$

We must therefore consider

$$I_4 = \int_1^{\infty} ((x)) \coth(x \log r) \frac{C \sinh(x \log r)}{e^{C \sinh(x \log r)} - 1} dx ,$$

$$I_5 = \int_1^{\infty} \log(1 - e^{-C \sinh(x \log r)}) dx .$$

Let us consider I_4 first.

Let

$$(3.2.17) \quad h(x) = - \sum_{v=-\infty}^{\infty} \frac{e^{2\pi i v x}}{4\pi v^2} + \sum_{\ell=1}^{\infty} \sum_{v=-\infty}^{\infty} \frac{e^{[2\pi i v - 2\ell \log r]x}}{2\pi i v (\pi i v - \ell \log r)} .$$

Then

(3.2.18)

$$I_4 = -h(2) \frac{C \cosh (\log r)}{e^C \cosh (\log r) - 1} - \int_1^\infty h(x) \frac{\partial}{\partial x} \left\{ \frac{C \sinh x \log r}{e^C \sinh x \log r - 1} \right\} dx$$

Now

$$\begin{aligned} & \int_1^\infty h(x) \frac{\partial}{\partial x} \left(\frac{C \sinh x \log r}{e^C \sinh (x \log r) - 1} \right) dx = \\ &= \frac{1}{\log r} \int_{\sinh \log r}^\infty h\left(\frac{\sinh^{-1} z}{\log r}\right) \frac{\partial}{\partial z} \frac{Cz}{e^{Cz} - 1} dz \\ &= \frac{1}{2\pi i \log r} \int_{\beta-i\infty}^{\beta+i\infty} \Gamma(t) \{(t-1)\zeta(t)\} C^{1-t} \\ & \quad \int_{\sinh \log r}^\infty h\left(\frac{\sinh^{-1} z}{\log r}\right) z^{-t} dz \quad (1 < \beta \leq 2) \end{aligned}$$

Since $h(x)$ is bounded the inversion is justified by the same arguments for I_3 .

Let us consider

$$\begin{aligned} v_1(t) &= \int_{\sinh (\log r)}^\infty h\left(\frac{\sinh^{-1} z}{\log r}\right) z^{-t} dz = \\ &= \log r \int_1^\infty h(x) [\sinh (u \log r)]^{-t} \cosh (u \log r) du \quad . \end{aligned}$$

Now by (3.2.17) $h(x) = \sum_{\ell=0}^\infty h_\ell(x)$, where

$$h_\ell(u) = \sum_{v=-\infty}^\infty \frac{e^{[2\pi i v - 2\ell \log r] u}}{2\pi i v [\pi i v - \ell \log r]} \quad .$$

Since we have uniform convergence and since $\operatorname{Re} t > 1$ we may integrate term-by-term. We thus obtain, following the same steps as in the treatment of I_3 , that

$$\log r \int_1^\infty h_\ell(x) [\sinh(u \log r)]^{-t} \cosh(u \log r) du$$

is analytic in the entire plane with simple poles at

$$t_{\ell,m}^v = 1 - 2\ell - 2m + \frac{2\pi i v}{\log r}, \quad \begin{array}{l} m = 0, 1, \dots \\ v = \pm 1, \pm 2, \dots \end{array}$$

with residues $2^{t-1} (2\pi i v)^{-1} (\pi i v - (\ell+m) \log r)^{-1}$ at $t_{\ell,m}^v$.

We thus obtain that $v_1(t)$ is analytic in the entire plane and its only poles with $\operatorname{Re} t > 0$ are simple and located at $1 + \frac{2\pi i v}{\log r}$ with residue $-2^{\frac{2\pi i v}{\log r}} (4\pi^2 v^2)^{-1}$.

We estimate

$$\frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \Gamma(t) \{(t-1)\zeta(t)\} C^{1-t} v_1(t) dt$$

by moving the path of integration from $\operatorname{Re} t = \beta$ to $\operatorname{Re} t = +\frac{1}{2}$. We obtain by arguments analogous to those used for $v(z)$ in the estimation of I_3 that along the line $\operatorname{Re} t = \beta$, $v_1(t) = O(1)$, along the line $\operatorname{Re} t = \beta$, $v_1(t) = O(I_n t)^2$, along the lines $I_m t = I(2N+1)\pi/\log r$ $v_1(t) = O(I_m t)^2$. We thus obtain

$$(3.2.19) \quad I_4 = -h(1) + \psi_2 \left(\log_r \frac{1}{\alpha} \right) + O(\alpha^{\frac{1}{2}}),$$

where ψ_2 is periodic of period one and has Fourier coefficients

$$c_v = - \frac{2^{\pi i v / \log r}}{4\pi^2 v^2} \Gamma \left(1 + \frac{2\pi i v}{\log r} \right) \frac{2\pi i v}{\log r} \zeta \left(1 + \frac{2\pi i v}{\log r} \right) 5^{-\frac{\pi i v}{\log r}}.$$

Now let us consider I_5 . We readily see that

$$\begin{aligned} I_5 &= \int_1^\infty \log (1 - e^{-C \sinh(x \log r)}) dx = \\ &= \frac{1}{\log r} \int_{C \sinh}^\infty \frac{\log(1 - e^{-z})}{\sqrt{z^2 + C^2}} (\log r) dz. \end{aligned}$$

We treat this integral similarly to I_1 . That is

$$\begin{aligned} (3.2.20) \quad \int_{C \sinh}^\infty \frac{\log(1 - e^{-z})}{\sqrt{z^2 + C^2}} dz &= \int_{C \sinh}^{C \frac{3}{8}} \frac{\log(1 - e^{-z})}{\sqrt{z^2 + C^2}} dz \\ &+ \int_{C \frac{3}{8}}^\infty \frac{\log(1 - e^{-z})}{z} dz + O \left\{ C^{\frac{3}{2}} \int_{\frac{3}{C8}}^\infty \frac{\log(1 - e^{-z})}{z} dz \right\}. \end{aligned}$$

Now

$$\int_\beta^\infty \frac{\log(1 - e^{-y})}{y} dy = -\log(1 - e^{-\beta}) \log \beta - \int_\beta^\infty \frac{\log y}{e^y - 1} dy,$$

and since

$$\begin{aligned}
 \int_{\beta}^{\infty} \frac{\log y}{e^y - 1} dy &= \int_{\beta}^{\infty} \frac{\log (1 - e^{-y})}{e^y - 1} dy + \int_{\beta}^{\infty} \frac{\log y - \log (1 - e^{-y})}{e^y - 1} dy \\
 &= -\frac{1}{2} \log^2 (1 - e^{-\beta}) + \int_0^{\infty} \frac{\log y - \log (1 - e^{-y})}{e^y - 1} dy + \\
 &\quad - \int_0^{\beta} \frac{\log y - \log (1 - e^{-y})}{e^y - 1} dy .
 \end{aligned}$$

Now the integrand in the last integral may be expanded in powers of y when $\beta = C^{3/8}$. We then obtain

$$\begin{aligned}
 (3.2.21) \quad \int_{C^{3/8}}^{\infty} \frac{\log (1 - e^{-z})}{z} &= -\frac{9}{128} \log^2 C - \int_0^{\infty} \frac{\log y - \log (1 - e^{-y})}{e^y - 1} dy \\
 &\quad + O(C^{3/16}) .
 \end{aligned}$$

To evaluate

$$\int_{C^{3/8}}^{\infty} \frac{\log z}{\sinh (\log r) \sqrt{z^2 + C^2}} dz$$

note that

$$(3.2.22) \quad \int_{\beta}^{\gamma} \frac{\log z}{\sqrt{z^2 + C^2}} = \sinh^{-1} \frac{z}{C} \log z \Big|_{\gamma}^{\beta} - \int_{\gamma}^{\beta} \frac{\sinh^{-1} \frac{z}{C}}{z} dz .$$

Also

$$(3.2.23) \quad \int_{\gamma}^{\beta} \frac{\sinh^{-1} \frac{z}{C}}{z} dx = \int_{\gamma}^{\beta} \frac{\log z}{z} + \int_{\gamma}^{\beta} \frac{\log (1 + \sqrt{1 + \frac{C^2}{z^2}})}{z} dz ,$$

and

$$(3.2.24) \quad \int_{\gamma}^{\beta} \frac{\log (1 + \sqrt{1 + \frac{C^2}{z^2}})}{z} dz = \int_{\gamma}^{\beta} \frac{\log 2}{z} + \int_{\gamma}^{\beta} \frac{\log (1 + \sqrt{1 + \frac{z^2}{C^2}}) - \log 2}{z} dz$$

Furthermore

$$\begin{aligned} (3.2.25) \quad & \int_{\gamma}^{\beta} \frac{\log (1 + \sqrt{1 + \frac{C^2}{z^2}}) - \log 2}{z} dz = \\ & = \int_{\gamma/C}^{\beta/C} \frac{\log (1 + \sqrt{1 + \frac{1}{\omega^2}}) - \log 2}{\omega} d\omega \\ & = \int_{\gamma/C}^{\infty} \frac{\log (1 + \sqrt{1 + \frac{1}{\omega^2}}) - \log 2}{\omega} d\omega \\ & - \int_{\beta/C}^{\infty} \frac{\log 2 + o(\frac{1}{\omega^2}) - \log 2}{\omega} d\omega \\ & = \int_{\gamma/C}^{\infty} \frac{\log (1 + \sqrt{1 + \frac{1}{\omega^2}}) - \log 2}{\omega} d\omega + o(C/\beta)^2 . \end{aligned}$$

Employing (3.2.20), (3.2.21), (3.2.22), (3.2.23), (3.2.24) and (3.2.25)

we obtain $(\beta = C^{3/8}, \gamma = C \sinh \log r)$.

$$\begin{aligned} (3.2.26) \quad & \int_{C \sinh (\log r)}^{\infty} \frac{\log (1 - e^{-z})}{\sqrt{z^2 + C^2}} dz = -\frac{1}{2} \log^2 C + \\ & + \log (\cosh \log r + \sinh \log r) + \end{aligned}$$

$$\begin{aligned}
 & - \log(\sinh \log r) \log(\sinh \log r + \cosh \log r) + \frac{1}{2}(\sinh(\log r))^2 + \\
 & - \int_{\sinh(\log r)}^{\infty} \frac{\log[(1 + \sqrt{1 + \omega^{-2}})/2]}{\omega} d\omega - \int_0^{\infty} \frac{\log y - \log(1 - e^{-y})}{e^y - 1} dy \\
 & + O\left(\frac{1}{C^8}\right) .
 \end{aligned}$$

From (3.2.26), (3.2.19), and (3.2.16) we obtain

(3.2.27)

$$\begin{aligned}
 \log F(\alpha) = & \frac{1}{2 \log r} \log^2 \frac{1}{\alpha} + \frac{1}{\log r} \log \left[\frac{\sqrt{5}}{\sinh \log r + \cosh \log r} \log \frac{1}{\alpha} + \right. \\
 & - \frac{1}{\log r} \log (\sinh \log r) \log(\sinh \log r + \cosh \log r) + \\
 & + \frac{\sinh^2(\log r)}{2 \log r} - \frac{1}{\log r} \int_{\sinh(\log r)}^{\infty} \frac{\log[(1 + \sqrt{1 + \omega^{-2}})/2]}{\omega} d\omega \\
 & + \frac{\log^2 \sqrt{5}/2}{\log r} - \frac{\sqrt{5}}{\log r} \log \left[\frac{\sqrt{5}}{\cosh \log r + \sinh \log r} \right] \log \frac{\sqrt{5}}{2} + \\
 & - \frac{1}{\log r} \int_0^{\infty} \frac{\log y - \log(1 - e^{-y})}{e^y - 1} dy + \psi_2 \left(\log_r \frac{1}{\alpha} \right) + \frac{1}{2} \sum_{v=1}^{\infty} \frac{1}{v} \\
 & - \sum_{\ell=1}^{\infty} \sum_{v=-\infty}^{\infty} \frac{e^{-2\ell \log r}}{2\pi i v (2\pi i v - \ell \log r)} + O\left(\alpha^{\frac{1}{2}}\right) .
 \end{aligned}$$

We still have to determine $\log A_2 - \frac{1}{2}$. In this case

$$A_2 = \frac{1}{\alpha^2} \sum_{n=2}^{\infty} C^2 \sinh^2 (n \log r) \frac{e^{C \sinh n \log r}}{(e^{C \sinh n \log r} - 1)^2}$$

This integral is treated as I_1 and I_4 were. The details are very similar to those in the treatment of I_1 . The conclusion is

(3.2.28)

$$-\frac{1}{2} \log A_2 = \log \frac{1}{\alpha} + \frac{1}{2} \log \log \frac{1}{\alpha} - \log [\sinh \log r + \cosh \log r] + O(\log^{-1} \frac{1}{\alpha}).$$

We have not yet shown that the functions ψ_1 and ψ_2 are uniformly continuous. This is easy however because it follows at once from (3.2.11) that the Fourier series involved are uniformly convergent. Since $e^{2\pi i \nu x}$ is a uniformly continuous function so must ψ_1 and ψ_2 be. Thus we can solve for $P_A(n)$ from (3.2.28), (3.2.27) and (3.2.15) in the same manner as we did in Mahler's problem.

We thus conclude that when A is defined by $a_n = F_{2n}$, where F_n is the n -th Fibonacci number

$$\begin{aligned} (3.2.29) \quad \log P_A(n) &= \frac{1}{2 \log r} \left[\log \left(\frac{n}{\log n} \right)^2 + \right. \\ &+ \left(\frac{\log \log r}{\log r} + \frac{1}{\log r} - 1 + \log [\sqrt{5} / \sinh \log r + \cosh \log r] \right) \log n + \\ &+ \left(\frac{3}{2} - \frac{\log \log r}{\log r} \right) \log \log n + \phi \left(\frac{\log n - \log \log n + \log \log r}{\log r} \right) \\ &\left. + O \left\{ \frac{(\log \log n)^2}{\log n} \right\} \right] \end{aligned}$$

where

$$\phi(x) = \sum_{\nu=-\infty}^{\infty} c_{\nu} e^{2\pi i \nu x}, \quad r = (1 + \sqrt{5})^2$$

and

$$c_v = \left(\frac{2}{5}\right)^{\frac{\pi i v}{\log r}} \Gamma\left(\frac{2\pi i v}{\log r}\right) \zeta\left(1 + \frac{2\pi i v}{\log r}\right) \quad v = \pm 1, \pm 2, \dots$$

and where

$$\begin{aligned} c_o = & \frac{\log \log r}{2 \log r} - \frac{\log \log r}{\log r} \log [\sqrt{5} / (\sinh \log r + \cosh \log r)] + \\ & - \log(\sinh \log r) \log(\sinh \log r + \cosh \log r) + \frac{\sinh^2(\log r)}{2 \log r} + \\ & - \frac{1}{\log r} \int_{\sinh(\log r)}^{\infty} \frac{\log[(1 + \sqrt{1 + \omega^{-2}})/2]}{\omega} d\omega + \frac{\log^2 \frac{\sqrt{5}}{2}}{\log r} + \\ & - \frac{\sqrt{5}}{\log r} \log [\sqrt{5} / (\sinh \log r + \cosh \log r)] \log(\sqrt{5}/2) + \frac{\pi}{12} + \\ & - \frac{1}{2} \log 2\pi - \frac{1}{\log r} \int_0^{\infty} \frac{\log y - \log(1 - e^{-y})}{e^y - 1} dy + \\ & - \sum_{\ell=1}^{\infty} \sum_{v=-\infty}^{\infty} \frac{e^{-2\ell \log r}}{(2\pi i v)(2\pi i v - \ell \log r)} . \end{aligned}$$

(3.3.30) Remark: If

$$a_v = \frac{r^{+v} - r^{-v}}{d}$$

then $\sqrt{5}$ must be replaced by d in (3.2.29). It is clear that a similar result can be derived when

$$a_v = \frac{r^v + r^{-v}}{d}$$

with $\sinh$ and $\cosh$ interchanged.

BIBLIOGRAPHY

- [1] E. Grosswald, The Theory of Numbers, The Macmillan Company, 1966).
- [2] G.H. Hardy and S. Ramanujan, Asymptotic formulae in combinatory analysis, Proc. London Math. Soc. (2), 17, (1918), pp. 75-115.
- [3] G.H. Hardy and S. Ramanujan, Asymptotic formulae for the distribution of integers of various types. Proc. London Math. Soc. (2), 16 (1917), 112-132.
- [4] N. Brigham, A general asymptotic formula for partition functions. Proc. Amer. Math. Soc. 1 (1950), 182-191.
- [5] E.E. Kohbecker, Weak asymptotic properties of partitions. Transactions Amer. Math. Soc. 88 (1958), 346-365.
- [6] S. Parameswaren, Partition functions, whose logarithms are slowly oscillating. Trans. Amer. Math. Soc. 100 (1961), 217-240.
- [7] W. Schwarz, Schwache asymptotic Eigenschaften von Partionen. J.f. reine u. angew. Math., 232 (1968), (1-16).
- [8] E. Wagner, Tauber'sch Sätze reeller Art für die Laplace-Transformation. Math. Nachr. 31 (1966), 153-168.
- [9] P. Erdős. On an elementary proof of some asymptotic formulas in the theory of partitions. Annals of Math. (2) 43 (1942), 437-450.
- [10] A.E. Ingham, A Tauberian theorem for partitions, Annals of Mathematics 42(1941), 1075-1090.

- [11] W.B. Pennington, On Mahler's partition problem. *Annals of Math.* (2) 57 (1953), 531-546.
- [12] W. Schwarz, Asymptotische Formeln für Partionen, *J.f. reine a. angew. Math.* 234 (1969), 172-178.
- [13] G. Meinardus, Asymptotic Aussagen über Partionen, *M.Z.* 59 (1954), 388-398.
- [14] K.F. Roth and G. Szekeres, Some asymptotic formulae in the theory of partitions. *Quart. J. Math. Oxford* (2) 5 (1954), 244-259.
- [15] K. Mahler, On a special functional equation. *J. London Math. Soc.* 15 (1940), 115-123.
- [16] N.G. de Bruijn, On Mahler's partition problem, *Indag. Math.* 10 (1943), 210-220.
- [17] W. Schwarz, C. Mahler's Partitionsproblem, *J. reine angew. Math.* 229 (1967), 182-188.
- [18] E.C. Titchmarsh, *The Theory of the Riemann Zeta-function.* (Oxford 1951).
- [19] E.C. Titchmarsh, *The Theory of Functions*, 2nd edition (Oxford, 1939).
- [20] W.H. Young, On the Integration of Fourier Series, *Proc. London Math. Soc.* 2 (1910), 449-426.
- [21] A. Zygmund, *Trigonometrical series* (Warsaw, 1935).
- [22] G. Pólya and G. Szegő, *Aufgaben und Lehrsätze aus der Analysis*, Vol. 1, Part 1, Problem 27 (Berline, Springer, 1925).
- [23] P.T. Bateman and P. Erdős, Monotonicity of Partition Functions, *Mathematika*, 3 (1956), 1-14.

- [24] P. Bachmann, *Niedere Zahlentheorie: Zweiter Teil, Additive Zahlentheorie*, Ch. 3 (Leipzig, Teubner, 1910).
- [25] E. Netto, *Lehrbuch der Combinatorik*, Ch. 6 (Leipzig, Teubner, 1901 and 1927).

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